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Turbulent resistivity in wavy two-dimensional magnetohydrodynamic turbulence

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The theory of turbulent resistivity in 'wavy' magnetohydrodynamic turbulence in two dimensions is presented. The goal is to explore the theory of quenching of turbulent resistivity in a regime for which the mean field theory can be rigorously constructed at large magnetic Reynolds number Rm. This is achieved by extending the simple two-dimensional problem to include body forces, such as buoyancy or the Coriolis force, which convert large-scale eddies into weakly interacting dispersive waves. The turbulence-driven spatial flux of magnetic potential is calculated to fourth order in wave slope – the same order to which one usually works in wave kinetics. However, spatial transport, rather than spectral transfer, is the object here. Remarkably, adding an additional restoring force to the already tightly constrained system of high Rm magnetohydrodynamic turbulence in two dimensions can actually increase the turbulent resistivity, by admitting a spatial flux of magnetic potential which is not quenched at large Rm, although it is restricted by the conditions of applicability of weak turbulence theory. The absence of Rm-dependent quenching in this wave-interaction-driven flux is a consequence of the presence of irreversibility due to resonant nonlinear three-wave interactions, which are independent of collisional resistivity. The broader implications of this result for the theory of mean field electrodynamics are discussed.

1. Introduction

Hydromagnetic turbulence is a ubiquitous feature of electrically conducting geophysical astrophysical and laboratory flows, and the dynamics of transport and amplification of magnetic fields in turbulent magnetofluids are topics of considerable interest. Hence, the desire to understand and predict the behaviour of large-scale magnetic fields in turbulence has driven the development of the theory of mean field electrodynamics (Steenbeck, Krause & Rädler 1966; Moffatt 1978). This theory seeks to calculate the evolution of the mean magnetic field $\langle \boldsymbol{B} \rangle$ in terms of the structure of the spectrum of ambient turbulence. In all cases, the actual construction of a tractable mean field equation requires some sort of quasi-linear closure of the averaged induction equation, in which the magnetic fluctuation $\boldsymbol{b}$ (i.e. the small-scale magnetic field) is eliminated in favour of the hydrodynamic fluctuation $\boldsymbol{v}$, the mean field $\langle \boldsymbol{B} \rangle$, and some response or correlation time τ_c. Typical products of the mean field electrodynamics industry are the turbulent resistivity η_T and the pseudo-scalar dynamo coefficient α, which parameterize the relation between the turbulent electromotive force $\langle \boldsymbol{v} \times \boldsymbol{b} \rangle$ and the mean field $\langle \boldsymbol{B} \rangle$ and associated current $\langle \boldsymbol{J} \rangle$, according to the well-known relation

$$\langle \boldsymbol{v} \times \boldsymbol{b} \rangle = \alpha \langle \boldsymbol{B} \rangle + \eta_T \langle \boldsymbol{J} \rangle. \tag{1.1}$$

Although mean field electrodynamics raises many questions, by far the most awkward are those concerned with the detailed physics of the correlation time τ_c, which relates the response of $\boldsymbol{b}$ to $\boldsymbol{v}$ and $\langle \boldsymbol{B} \rangle$. This is because magnetohydrodynamic (MHD) turbulence is, by its nature, a complex dynamical system in which two fluid fields, $\boldsymbol{v}(\boldsymbol{x}, t)$ and $\boldsymbol{b}(\boldsymbol{x}, t)$, evolve nonlinearly under each other's influence. At the same time, the magnetic field is frozen into the fluid field, except on small scales, where the collisional resistivity η_c allows $\boldsymbol{b}$ to slip relative to $\boldsymbol{v}$. This freezing-in law, which follows from Alfvén's theorem, places an especially severe constraint on any hypothesized irreversible process in MHD.

In particular, the freezing-in law suggests that the cross-phase, $\langle v_x A \rangle$, which is equivalent to the spatially averaged vertical flux of magnetic potential A, should depend explicitly upon the collisional resistivity – a result due to Zel'dovich (1957). This is not good news for MHD relaxation processes or dynamos at high magnetic Reynolds number $Rm = \mathscr{U}\ell/\eta_c$ (where $\mathscr{U}$ and ℓ are the typical eddy velocity and length scale) since resistive diffusion rates are far too slow to be of any practical interest. In view of the enormous values of Rm found in astrophysical plasmas (10^7 in stellar convection zones), this suppression is sometimes termed 'catastrophic quenching', and places a limit on observable flux production in cosmic dynamos. In the past, this issue has been side-stepped by invoking cascades, which couple inertially dominated large scales to dissipative small scales, as a means of justifying the linking of τ_c to the turn-over time for large eddies.

Undoubtedly the most basic problem in mean field electrodynamics is the calculation of turbulent diffusivity of mean magnetic potential in two-dimensional MHD. Recall that, in incompressible two-dimensional MHD, the freezing-in law is simply the statement that magnetic potential A is conserved along a fluid element trajectory, up to resistive diffusion. In retrospect, then, it is no surprise that a substantive debate on the issue of the fundamental underpinnings of irreversibility in mean field electrodynamics first surfaced with the seminal work of Cattaneo & Vaĭnshteĭn (1991) in which it was shown that, in two-dimensional MHD, the turbulent resistivity is severely quenched below the kinematic prediction. By a combination of numerical calculations and physical reasoning, it was demonstrated that the actual mean field resistivity η_T is related to the kinematic mean field resistivity η_{kin} by the relation

$$\eta_T = \frac{\eta_{kin}}{1 + Rm\, V_{A0}^2/\langle v^2 \rangle}. \tag{1.2}$$

Here, $\eta_{kin} \approx \langle v^2 \rangle \tau_c$, where τ_c is a large-eddy turn-over time, and $V_{A0}^2 = \langle B \rangle^2 / 4\pi\rho_0$ and $\langle v^2 \rangle$ is the square of the Alfvén velocity. Note that even for weakly magnetized flows with $Rm\, V_{A0}^2/\langle v^2 \rangle > 1$, the turbulent resistivity at high Rm is quenched to the level of the collisional resistivity η_c enhanced by the factor $\langle v^2 \rangle / V_{A0}^2$. Alternatively put, the explicit proportionality of η_T to η_c is inescapable at high Rm, irrespective of the presence of cascades, multi-scale couplings, etc. Thus, these findings suggested that the MHD freezing-in law directly regulates and limits mean field transport.

The original computational work by Cattaneo & Vaĭnshteĭn (1991) studied turbulent diffusion of mean magnetic field by examining macroscopic evolution, namely, by following the relaxation of large-scale field gradients in a two-dimensional periodic box. While their study was for relatively modest Rm, later studies of two-dimensional hydromagnetic turbulence in similar configurations supported the prediction of an Rm-dependent quench (Cattaneo 1994; Silvers 2005). Subsequent work has been concerned with the theoretical understanding and interpretation of quenching (discussed in detail below), the Rm-dependent quenching of α in three-dimensional helical turbulence (Kleeorin & Ruzmaikin 1982; Zel'dovich, Ruzmaikin & Sokoloff 1983; Kulsrud & Anderson 1992; Gruzinov & Diamond 1994), and with extensions to systems with more geometric complexity, e.g. open boundaries, inhomogeneities, etc. (Blackman & Field 2000). Despite the plethora of subsequent publications stimulated by the seminal work of Cattaneo & Vaĭnshteĭn (1991), relatively little attention has been devoted to understanding the detailed physics behind the phenomenon of resistivity quenching. Concerns about boundaries, inhomogeneity, and α-quenching notwithstanding, it is to the question of the fundamental physics of η_T-quenching (in the context of the most basic possible problem) which we now turn.

1.1. *The theory of turbulent resistivity quenching in two-dimensional MHD turbulence*

The literature on this subject is reviewed in Diamond *et al.* (2005). A brief survey of the current understanding of η_T-quenching might profitably be structured as a progression through increasingly fine-grained descriptions of the spatial transport of magnetic potential A via:

(i) global magnetic potential balance and Zel'dovich's theorem;
(ii) competing couplings in the transfer of $\langle A^2\rangle_{\boldsymbol{k}}$;
(iii) closure calculations of the flux transport.

At the simplest level (i), the global balance of A^2 – which is conserved in two-dimensional MHD up to dissipation and boundary fluxes – yields

$$\eta_T = \eta_c \frac{\langle b^2\rangle}{\langle B\rangle^2}. \tag{1.3}$$

Equation (1.3) reproduces the well-known Zel'dovich (1957) relation $\langle b^2\rangle \approx Rm\langle B\rangle^2$ when $\eta_T/\eta_c \approx Rm$, as is generally assumed (Cattaneo & Vaĭnshteĭn 1991; Diamond *et al.* 2005). Thus, if $\langle b^2\rangle$ is independent of η_c, then $\eta_T \propto \eta_c$, suggestive of quenching. Indeed, if $\langle b^2\rangle \approx \langle v^2\rangle$, then η_T is exactly that predicted by Cattaneo & Vaĭnshteĭn (1991) in the limit $Rm V_{A0}^2/\langle v^2\rangle > 1$. Of course, for $\langle b^2\rangle$ to be independent of η_c requires that the magnetic fluctuation spectrum decreases sufficiently rapidly with k, i.e. as $k^{-\alpha}$, with $\alpha > 1$. While two-dimensional unmagnetized MHD turbulence (i.e turbulence for which $\langle b^2\rangle \gg \langle B\rangle^2$) is known to have $\alpha \approx 3/2$, the scaling exponent α is not known for more general circumstances, and so the independence of $\langle b^2\rangle$ from η_c must be regarded as an assumption, the validity of which remains to be demonstrated.

To understand the arguments for quenching at the second level (ii), it is useful to consider the fate of a group of isocontours (i.e. loops) of magnetic potential in a turbulent two-dimensional flow. In the absence of $\boldsymbol{j}\times\boldsymbol{b}$ forces – the kinematic case – the magnetic potential is advected as a passive scalar. The flux loops are strained apart by the turbulent flow, a process which is characterized by a forward cascade of A^2 toward small scales. On the other hand, it is well known that current filaments of like sign attract each other. This process drives the coalescence of flux loops onto progressively larger and larger scales, and so is characterized by an inverse cascade of A^2 (Fyfe & Montgomery 1976; Fyfe, Montgomery & Joyce 1977). Note that the forward cascade dominates if $\langle v^2\rangle \gg \langle b^2\rangle$ (so that the field is passively advected by the flow), and the inverse cascade dominates if $\langle b^2\rangle \gg \langle v^2\rangle$. The forward cascade is parameterized by a positive turbulent resistivity, while the inverse cascade is an example of a negative viscosity phenomenon. Thus, these two processes are natural competitors, and, in fact, cancel each other for near-Alfvénic turbulence, greatly suppressing η_T. That is, η_T is greatly suppressed when the turbulent fluctuations are at or near equipartition: $\langle v^2\rangle = \langle b^2\rangle$. (This need not be the case, however: see Ting, Montgomery & Matthaeus 1986). The quenched η_T, i.e. from (1.2) with $Rm V_{A0}^2/\langle v^2\rangle > 1$, is identical to the η_T

derived from considerations of quasi-linear diffusion of magnetic flux by a spectrum of Alfvén waves, as discussed in Appendix B. This struggle to a stand-off between competing cascades of $\langle A^2\rangle$ offers a second description of the origin of turbulent resistivity quenching in two dimensions. In contrast to the Zel'dovich relation (1.3), it is local and linked to MHD physics, albeit qualitatively.

Looking further into turbulence theory, we can undertake an eddy-damped quasi-normal Markovian (EDQNM) calculation of the flux of magnetic potential – a route followed by Gruzinov & Diamond (1994, 1996), building upon earlier work by Pouquet, Frisch & Leorat (1976) and Pouquet (1978). A quasi-linear closure calculation of the flux of magnetic potential $\Gamma_A = \langle v_z A\rangle$ follows the usual recipe, i.e.

$$\begin{aligned}\Gamma_A &= \langle v_x \delta A\rangle + \langle A \delta v_z\rangle \\ &= \mathrm{Re}\sum_{\boldsymbol{k}} \mathrm{i}k_x(\tilde{A}_{-\boldsymbol{k}}\delta\psi_{\boldsymbol{k}} - \psi_{-\boldsymbol{k}}\delta\tilde{A}_{\boldsymbol{k}}),\end{aligned} \tag{1.4}$$

where $\delta\psi_{\boldsymbol{k}}$ and $\delta A_{\boldsymbol{k}}$, the quasi-linear responses of the fluid and the field to $\langle B\rangle$ (assumed to be pointing in the x-direction), are given by

$$\frac{\delta\psi_{\boldsymbol{k}}}{\tau_c} = \mathrm{i}\langle B\rangle k_x A_{\boldsymbol{k}}, \quad \frac{\delta\tilde{A}_{\boldsymbol{k}}}{\tau_c} = \mathrm{i}\langle B\rangle k_x \psi_{\boldsymbol{k}}. \tag{1.5a,b}$$

Notice the appearance, once again, of the correlation time τ_c. Straightforward substitution then gives the turbulent resistivity as

$$\eta_T = \frac{\Gamma_A}{\langle B\rangle} = \sum_{\boldsymbol{k}} \tau_c(\langle v^2\rangle_{\boldsymbol{k}} - \langle b^2\rangle_{\boldsymbol{k}}). \tag{1.6}$$

This result crystallizes the intuition, discussed previously, that η_T-quenching results from a competition between forward and inverse cascades, or, equivalently, between positive and negative viscosity effects. Substitution of (1.3) into (1.6) then recovers (1.2), the result of Cattaneo & Vaĭnshteĭn (1991) for resistivity quenching.

While the theory discussed here does give insight into the physics of turbulent resistivity quenching, and seems to agree with the results of numerical simulations, it is intrinsically unsatisfactory. In particular, τ_c is assumed to correspond to an eddy turn-over time – a crude simplification – and possible dependencies on $\boldsymbol{k}$, magnetic Reynolds number and magnetic Prandtl number are ignored. More fundamentally, these calculations offer little insight into the physical origin of the small-scale irreversibility and its effects on turbulent diffusion of magnetic fields. Indeed, the microphysics only enter the dynamics via an *ad hoc* model of turbulent mixing, parameterized by τ_c. Attempts to bolster or justify this approach, such as the 'τ-approximation' (Kleeorin, Rogachevskii & Ruzmaikin 1990) and the 'minimal τ-approximation' (Blackman & Field 2002) have done no better in addressing these fundamental questions. Moreover, while numerical simulations have compared the scalings of the measured macroscopic turbulent resistivity to those of the theoretical predictions, they have in no way stressed, probed, or validated the essentials of the theory – in particular, the microphysics of τ_c, which is, in essence, a free parameter in EDQNM calculations. A recent exception to this trend has observed a relation between η_T-quenching and the behaviour of the cross-phase between velocity and magnetic potential (Silvers, Keating & Diamond 2008).

Given this state of affairs, it is natural to explore extensions of the theory which might constrain the assumptions made concerning the small-scale correlation time τ_c. Therefore, in this paper we explore the transport of magnetic potential in high

Rm ‘wavy’ MHD turbulence in two dimensions. By ‘wavy’ MHD turbulence we mean dispersive wave turbulence which results from the addition of body forces, such as buoyancy or the Coriolis force, to the usual equations of incompressible two-dimensional MHD, namely,

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\nabla^2\psi = (\boldsymbol{B}\cdot\nabla)\nabla^2 A + \mathscr{F}_{body} + \nu\nabla^2\nabla^2\psi, \tag{1.7a}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)A = \eta_c\nabla^2 A, \tag{1.7b}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{B} = 0, \tag{1.7c}$$

where the fluid velocity $\boldsymbol{v}$ and magnetic field $\boldsymbol{B}$ are described by a streamfunction ψ and magnetic potential A in the plane of motion perpendicular to the unit normal $\hat{\boldsymbol{n}}$:

$$\boldsymbol{v} = \nabla\psi\times\hat{\boldsymbol{n}}, \qquad \boldsymbol{B} = \nabla A\times\hat{\boldsymbol{n}}. \tag{1.8a, b}$$

As some body forces, notably buoyancy, arise from the presence of an additional advected scalar field χ, the system is closed by the continuity equation for this scalar field.

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\chi = \mathscr{D}\nabla^2\chi. \tag{1.9}$$

As we shall show, the addition of the body force $\mathscr{F}_{body}$ has the effect of converting large-scale eddys into dispersive waves, for example, magneto-internal waves or Rossby waves. Because *Rm* is large, the nonlinear dynamics are non-dissipative. Moreover, since the wave slope $k\tilde{\varepsilon}$ is smaller than unity (here $\tilde{\varepsilon}$ is a fluid element displacement), within at least some region of wavenumber space, the waves are weakly interacting, and so weak or wave turbulence theory is applicable.

Weak turbulence theory has several attractions which are of particular relevance to the questions at issue here. First, the origin of small-scale irreversibility in weak turbulence theory is in three-wave resonances, which are present, even in the dissipationless limit, via the Landau pole prescription, and usually enter the theory as $\delta(\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{k}'} - \omega_{\boldsymbol{k}''})$. These resonance functions identify those mode-interaction triads which make secular contributions to the energy transfer. Given the unambiguous nature of dissipationless energy transfer, the physics of τ_c is now clear, constrained, and set by the spectral auto-correlation time τ_{ac}. All that is required for the validity of weak turbulence theory, in addition to the overlap of three-wave resonances, is that $\tau_{ac} \ll \tau_{tr}$, where τ_{tr} is the energy transfer time. It is important to note that the condition for weak interactions (i.e. $k\tilde{\varepsilon} < 1$) does not imply that *Rm* is small: the wave-slope and magnetic Reynolds number are independent asymptotic parameters, and $k\tilde{\varepsilon}$ can remain small but finite even as $Rm \to \infty$. Thus, weak turbulence theory, though limited in its range of applicability, possesses many attributes which render it a valuable test bed for the study of turbulent resistivity quenching.

While weak turbulence theory has been applied to incompressible MHD turbulence in the past (Ng & Bhattacharjee 1997; Galtier *et al.* 2000), two factors distinguish the present work from previous studies. First, the theory of weakly interacting Alfvén waves in ideal incompressible MHD turbulence is quite different from classical weak turbulence theory, the subject of our investigation. As Galtier *et al.* (2000) point out, this is because Alfvén waves are non-dispersive – a property which ordinarily leads to a secular growth of co-moving interacting wavetrains, ultimately invalidating the ‘weakly interacting’ assumption upon which the theory is built. However, Alfvén waves possess the special property that co-moving wave packets do not interact, and so a weak turbulence theory can be constructed for incompressible MHD, albeit one which is considerably different from classical weak turbulence theory.

In this work, by contrast, we apply the classical theory of wave turbulence to a model of two-dimensional MHD turbulence with large-scale dispersive waves arising from the presence of non-ideal body forces. Therefore, the calculations presented herein are more akin to the work of Kaburaki & Uchida (1971) and Mikhailovskii *et al.* (1989). These studies examined Alfvén wave turbulence incorporating, respectively, compressibility and finite Larmor radius, giving rise to dispersive linear modes which could, in turn, be studied under the weak-interaction hypothesis.

The second factor which distinguishes this work is that, unlike the classical wave turbulence analyses (including Kaburaki & Uchida 1971; Mikhailovskii *et al.* 1989), we do not seek to obtain a kinetic equation describing the spectral transfer of energy among interacting modes. Rather, our goal here is to calculate the spatial transfer of magnetic potential directly, using a systematic expansion in powers of the turbulent amplitude to rigorously derive an expression for the turbulent resistivity in terms of the wave amplitudes.

Wavy MHD in two dimensions may be of more than purely academic interest, however. Hydromagnetic turbulence in astrophysical and laboratory contexts frequently exhibits non-Alfvénic wave modes. Moreover, the effects of rotation, stratification, or a strong axial field can significantly limit fluid motions in one direction. For example: a rapidly rotating magnetofluid can be modelled with the two-dimensional magnetohydrodynamic (MHD) equations (Bracco *et al.* 1998); solar tachocline turbulence is quasi-geostrophic MHD turbulence in a spherical shell (Diamond *et al.* 2007); and strong external fields can restrict motion along field lines (Tsinober 1975).

In this paper, then, we calculate the spatial flux of magnetic potential in wavy MHD, with the aim of studying turbulent resistivity quenching in a system where the physics of small-scale irreversibility (i.e. τ_c) is unambiguous and independent of the collisional resistivity η_c. We calculate Γ_A to fourth order in fluctuation level (or, equivalently, wave-slope). This is the same order to which one usually works when constructing a wave kinetic equation; however, in this case we calculate spatial transport rather than spectral transfer.

The contribution to the flux of magnetic potential from nonlinear wave interactions, and therefore the turbulent resistivity, may be expanded in powers of the wave amplitude:

$$\eta_T = \overset{(2)}{\eta_T} + \overset{(4)}{\eta_T} + \cdots . \tag{1.10}$$

Given that we are dealing with a system which possesses restoring forces in addition to those found in simple MHD, we might suspect that η_T would be more severely quenched than before. This is not the case, however. Rather, we find that the asymptotic behaviour of η_T for large Rm is

$$\eta_T = \overset{(2)}{\eta_T}\,(Rm^{-1}) + \overset{(4)}{\eta_T}\,(Rm^{0}) + \cdots . \tag{1.11}$$

The quasi-linear (second-order) part of η_T is strongly quenched with Rm, as the only irreversibility available at second order is collisional resistive diffusion ηk^2. By contrast, the fourth-order contribution is independent of Rm, since three-wave resonances provide an alternative means of irreversibility, even when $Rm \gg 1$. Hence, τ_c is unambiguous for this contribution, and directly linked to the spectral auto-correlation time.

Of course, the fourth-order contribution cannot be large, since it is intrinsically of order $(k\tilde{\varepsilon})^4$, and the wave slope $k\tilde{\varepsilon}$ is less than unity, as required for weak turbulence theory to apply. Nonetheless, the fourth-order flux, although small in wave amplitude, is not asymptotically small in Rm, as is the case for the quenched η_T in ordinary two-dimensional MHD turbulence. Thus, it seems fair to say that the presence of resonant wave interactions in two-dimensional MHD makes for an exception and challenge to our understanding of the Rm-dependent quench of η_T predicted by Cattaneo & Vaĭnshteĭn (1991).

The remainder of this paper is organized as follows: in § 2, we formulate the theory of wavy MHD in two dimensions and illustrate the formalism with examples incorporating the effects of rotation and stratification. In § 3, we indicate the existence of a regime of wavenumber-space where the turbulence is weak and three-wave resonances dominate the transfer of energy. This permits a well-defined expansion in powers of the turbulence intensity and the unambiguous interpretation of τ_c as the triad coherence time. The fourth-order contribution to the down-gradient flux of any advected scalar, magnetic potential or otherwise, is calculated explicitly, and we comment on some general properties of the result. We summarize and discuss our findings in § 4.

2. Wavy MHD in two dimensions

External influences, such as rotation and gravity, will modify the dynamics of two-dimensional MHD, at least within some regime of wavenumber space, even if profiles, stratification, etc. are such that all modes are linearly stable. Recognition of the importance of waves in modifying transport coefficients in MHD turbulence is not new, especially in the context of the dynamo problem: in particular, we mention the pioneering work of Moffatt (1970, 1972) for inertial waves, and Vaĭnshteĭn & Zel'dovich (1972) for sound waves. However, the main goal of these early studies was to use wave motion to imbue the turbulence with the helicity underlying the α-effect. By contrast, we shall show that the dispersive character of these modified waves permits us to clarify the physics of the correlation time τ_c, and, moreover, that wave interactions drive a flux of magnetic potential which is independent of the magnetic Reynolds number Rm.

This general statement can be extended to two-dimensional MHD turbulence in the presence of a number of dynamical scalar fields – what we have called 'wavy MHD'. All of the features of n-scalar wavy MHD are present in the two-scalar case, so without loss of generality, we admit a single scalar field $\chi(\boldsymbol{x}, t)$ in addition to the magnetic potential $A(\boldsymbol{x}, t)$. It is a straightforward exercise to extend the formulation to n scalars.

2.1. *Formulation of wavy MHD in two dimensions*

Consider a velocity field $\boldsymbol{v} = \nabla\psi \times \hat{\boldsymbol{y}}$ advecting the magnetic field $\boldsymbol{B} = \nabla A \times \hat{\boldsymbol{y}}$ and a scalar field $\chi(\boldsymbol{x}, t)$ – for instance, density, potential temperature, chemical concentration, etc. – in a square box with periodic boundary conditions. The potential $A(\boldsymbol{x}, t)$ reacts back upon the flow via the Lorentz force; the scalar $\chi(\boldsymbol{x}, t)$ can be advected passively or actively, depending on the form of the momentum equation.

As discussed in § 1, the equations of motion for the system have the form of the usual two-dimensional MHD equations with an additional body force:

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\nabla^2\psi = (\boldsymbol{B} \cdot \nabla)\nabla^2 A + \mathscr{F}_{body} + \nu\nabla^2\nabla^2\psi + \tilde{f}, \tag{2.1a}$$

$$(\partial_t + \boldsymbol{v} \cdot \nabla)A = \eta_c\nabla^2 A, \tag{2.1b}$$

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\chi = \mathscr{D}\nabla^2\chi, \tag{2.1c}$$

where ν is the molecular viscosity, η_c is the collisional resistivity, and $\mathscr{D}$ is the collisional diffusivity of the scalar χ. Any body forces in addition to the Lorentz force, such as the Coriolis force or buoyancy coupling, are contained in $\mathscr{F}_{body}$, which we leave unspecified for the time being. However, we consider only cases where the modes associated with the additional body forces are linearly stable. Finally, $\tilde{f}$ represents a random forcing term. We shall not consider $\tilde{f}$ explicitly here; rather, we assume that the ultimate effect of $\tilde{f}$ is to set the spectrum of fluctuations.

Let us assume that A and χ possess slowly varying mean gradients in the z-direction:

$$A(\boldsymbol{x}, t) = \langle A\rangle(z) + \tilde{A}(\boldsymbol{x}, t), \quad \sigma_1 = -\frac{\partial\langle A\rangle}{\partial z} \geqslant 0, \tag{2.2a,b}$$

$$\chi(\boldsymbol{x}, t) = \langle \chi\rangle(z) + \tilde{\chi}(\boldsymbol{x}, t), \quad \sigma_2 = -\frac{\partial\langle \chi\rangle}{\partial z} \geqslant 0. \tag{2.2c,d}$$

Separating the mean and fluctuating components in the equations of motion, dropping tildes where there is no ambiguity, and transforming to Fourier space yields equations of the general form

$$(\partial_t - 2\mathrm{i}f(\boldsymbol{k})k_x)\psi_{\boldsymbol{k}} - \mathrm{i}\sigma_1 k_x g_1^2(\boldsymbol{k})A_{\boldsymbol{k}} - \mathrm{i}\sigma_2 k_x g_2^2(\boldsymbol{k})\chi_{\boldsymbol{k}} = \mathscr{N}_{\boldsymbol{k}}^{(\psi)}, \tag{2.3a}$$

$$\partial_t A_{\boldsymbol{k}} - \mathrm{i}\sigma_1 k_x \psi_{\boldsymbol{k}} = \mathscr{N}_{\boldsymbol{k}}^{(A)}, \tag{2.3b}$$

$$\partial_t \chi_{\boldsymbol{k}} - \mathrm{i}\sigma_2 k_x \psi_{\boldsymbol{k}} = \mathscr{N}_{\boldsymbol{k}}^{(\chi)}, \tag{2.3c}$$

where

$$\mathscr{N}_{\boldsymbol{k}}^{(A)} = \tfrac{1}{2}\sum_{\boldsymbol{k}'+\boldsymbol{k}''=-\boldsymbol{k}} (\boldsymbol{k}'\cdot\boldsymbol{k}''\times\hat{\boldsymbol{y}})(A_{-\boldsymbol{k}'}\psi_{-\boldsymbol{k}''} - A_{-\boldsymbol{k}''}\psi_{-\boldsymbol{k}'}), \tag{2.4a}$$

$$\mathscr{N}_{\boldsymbol{k}}^{(\chi)} = \tfrac{1}{2}\sum_{\boldsymbol{k}'+\boldsymbol{k}''=-\boldsymbol{k}} (\boldsymbol{k}'\cdot\boldsymbol{k}''\times\hat{\boldsymbol{y}})(\chi_{-\boldsymbol{k}'}\psi_{-\boldsymbol{k}''} - \chi_{-\boldsymbol{k}'}A_{-\boldsymbol{k}''}), \tag{2.4b}$$

are the nonlinearities in the A and χ equations arising from the advection terms $\boldsymbol{v}\cdot\nabla A$ and $\boldsymbol{v}\cdot\nabla\chi$. The exact form of $\mathscr{N}_{\boldsymbol{k}}^{(\psi)}$, the nonlinearity in (2.3*a*), determines how χ reacts back upon the flow; $\mathscr{N}_{\boldsymbol{k}}^{(\psi)}$ is specified in turn by conservation laws, as we shall shortly discuss.

The functions $f(\boldsymbol{k})$, $g_1(\boldsymbol{k})$ and $g_2(\boldsymbol{k})$ are related to the dispersiveness of the linear modes. Reality of the wave-frequencies implies that $f(\boldsymbol{k})$, $g_1(\boldsymbol{k})$ and $g_2(\boldsymbol{k})$ are even functions of the wavevector $\boldsymbol{k}$. In the case of wavy MHD, $g_1 \equiv 1$ (and $\sigma_1 \equiv B_0$), as Alfvén waves obey the linear dispersion relation $\omega_{\boldsymbol{k}} = B_0 k_x$. It is convenient to allow arbitrary $g_1(\boldsymbol{k})$ for the time being, however. We demand that $g_2 \neq$ constant, so that the linear modes arising from the presence of the scalar field χ are strictly dispersive. The case of $g_2(\boldsymbol{k}) \equiv 0$ corresponds to passive advection of the scalar $\chi(\boldsymbol{x}, t)$.

To exploit the self-adjointness of the equations of motion, we introduce

$$C_{\boldsymbol{k}} = \begin{Bmatrix} g_1(\boldsymbol{k})A_{\boldsymbol{k}} \\ g_2(\boldsymbol{k})\chi_{\boldsymbol{k}} \end{Bmatrix}, \quad \Omega_{\boldsymbol{k}} = \begin{Bmatrix} \sigma_1 g_1(\boldsymbol{k})k_x \\ \sigma_2 g_2(\boldsymbol{k})k_x \end{Bmatrix}, \tag{2.5a,b}$$

$$\lambda_{\boldsymbol{k}} = f(\boldsymbol{k})k_x. \tag{2.5c}$$

We recognize $\Omega_{j,\boldsymbol{k}}$ as the linear frequency associated with the field $C_{j,\boldsymbol{k}}$. Under the transformations (2.5*a*)–(2.5*c*), the equations of motion become

$$\partial_t x_{\boldsymbol{k}}^a - \mathrm{i}\mathscr{L}_{\boldsymbol{k}}^{ab} x_{\boldsymbol{k}}^b = N_{\boldsymbol{k}}^a, \tag{2.6}$$

where $x_{\boldsymbol{k}}^a \in \{\psi_{\boldsymbol{k}}, C_{1\boldsymbol{k}}, C_{2\boldsymbol{k}}\}$. The linear operator is given by

$$\mathscr{L}_{\boldsymbol{k}} = \begin{pmatrix} 2\lambda_{\boldsymbol{k}} & \Omega_{\boldsymbol{k}}^{(1)} & \Omega_{\boldsymbol{k}}^{(2)} \\ \Omega_{\boldsymbol{k}}^{(1)} & 0 & 0 \\ \Omega_{\boldsymbol{k}}^{(2)} & 0 & 0 \end{pmatrix}. \tag{2.7}$$

Let us write the nonlinearities appearing in (2.6) in terms of an interaction matrix $\mathscr{M}_{\boldsymbol{kpq}}$:

$$N_{\boldsymbol{k}}^{(a)} = \tfrac{1}{2} \sum_{\boldsymbol{k}'+\boldsymbol{k}''=-\boldsymbol{k}} \mathscr{M}_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''}^{abc} x_{-\boldsymbol{k}'}^a x_{-\boldsymbol{k}''}^b. \tag{2.8}$$

From (2.4*a*), (2.4*b*) and (2.5*a*), the elements $\mathscr{M}_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''}^{\lambda bc}$ are, for the case $a = \lambda \in \{1, 2\}$ and $b, c \in \{\psi, 1, 2\}$,

$$\mathscr{M}_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''}^{\lambda bc} = (\boldsymbol{k}' \cdot \boldsymbol{k}'' \times \hat{\boldsymbol{y}}) \left\{ \frac{g_\lambda(\boldsymbol{k})}{g_\lambda(\boldsymbol{k}')} \delta^{b\lambda}\delta^{c\psi} - \frac{g_\lambda(\boldsymbol{k})}{g_\lambda(\boldsymbol{k}'')} \delta^{c\lambda}\delta^{b\psi} \right\}. \tag{2.9}$$

In Appendix A we show how energy conservation relates the remaining elements of $\mathscr{M}_{\boldsymbol{kpq}}$ to the functions g_1 and g_2:

$$\mathscr{M}_{\boldsymbol{kpq}}^{\psi\psi\psi} = (\boldsymbol{k} \cdot p \times \hat{\boldsymbol{y}}) \left\{ \frac{p^2 - q^2}{k^2} \right\}, \tag{2.10a}$$

$$\mathscr{M}_{\boldsymbol{kpq}}^{\psi ij} = -(\boldsymbol{k} \cdot p \times \hat{\boldsymbol{y}}) \left\{ \frac{p^2 g_i^2(\boldsymbol{p}) - q^2 g_i^2(\boldsymbol{q})}{k^2 g_i(\boldsymbol{p}) g_i(\boldsymbol{q})} \right\} \delta^{ij}. \tag{2.10b}$$

The appearance of the Kronecker delta in (2.10*b*) indicates that any dynamical scalar field appearing in the quadratic nonlinearity in the momentum equation acts only upon itself – no cross-terms appear. This is certainly the case for both ψ and A, appearing as they do in the form $\boldsymbol{v} \cdot \nabla \boldsymbol{v}$ and $\boldsymbol{b} \cdot \nabla \boldsymbol{b}$. Likewise, any χ-dependent nonlinearity must be of the form $\mathscr{F}(\chi, \chi)$. Also note that, in the special case of $g_2(\boldsymbol{k}) \propto k^{-1}$, the matrix element for the χ-nonlinearity vanishes identically. As we shall see, this is the case for two-dimensional Boussinesq flow of a magnetofluid in the presence of stratification. Finally, we note that, in the special case of $g_1 = g_2 = 1$, the coefficients $\mathscr{M}$ reduce to those obtained by Fyfe & Montgomery (1976) for ideal two-dimensional MHD.

2.2. *Linear theory*

The linear operator $\mathscr{L}_{\boldsymbol{k}}$ appearing in (2.6) has the following characteristic equation

$$(\omega - 2\lambda_{\boldsymbol{k}})\omega^2 - \Omega_{\boldsymbol{k}}^2 \omega = 0, \tag{2.11}$$

where $\Omega_{\boldsymbol{k}}^2 = \Omega_{1,\boldsymbol{k}}^2 + \Omega_{2,\boldsymbol{k}}^2$. The non-zero solutions of (2.11) are

$$\omega_{\boldsymbol{k}}^{(\pm)} = \lambda_{\boldsymbol{k}} \pm \sqrt{\lambda_{\boldsymbol{k}}^2 + \Omega_{\boldsymbol{k}}^2}, \tag{2.12}$$

with eigenvectors

$$\boldsymbol{v}_{\boldsymbol{k}}^{(+)} = \begin{Bmatrix} \cos a_{\boldsymbol{k}} \\ \alpha_{\boldsymbol{k}}^{(1)} \sin a_{\boldsymbol{k}} \\ \alpha_{\boldsymbol{k}}^{(2)} \sin a_{\boldsymbol{k}} \end{Bmatrix}, \quad \boldsymbol{v}_{\boldsymbol{k}}^{(-)} = \begin{Bmatrix} \sin a_{\boldsymbol{k}} \\ -\alpha_{\boldsymbol{k}}^{(1)} \cos a_{\boldsymbol{k}} \\ -\alpha_{\boldsymbol{k}}^{(2)} \cos a_{\boldsymbol{k}} \end{Bmatrix}, \tag{2.13a, b}$$

where

$$\alpha_{\boldsymbol{k}}^{(j)} = \frac{\Omega_{j,\boldsymbol{k}}}{\Omega_{\boldsymbol{k}}}, \qquad \sum_j \left(\alpha_{\boldsymbol{k}}^{(j)}\right)^2 = 1, \tag{2.14a,b}$$

and

$$\cos 2a_{\boldsymbol{k}} = \frac{\lambda_{\boldsymbol{k}}}{\sqrt{\lambda_{\boldsymbol{k}}^2 + \Omega_{\boldsymbol{k}}^2}}. \tag{2.15}$$

The eigenvector associated with $\omega \equiv 0$ can be written

$$\boldsymbol{v}_{\boldsymbol{k}}^{(0)} = \left\{0 \quad r_{\boldsymbol{k}}^{(1)} \quad r_{\boldsymbol{k}}^{(2)}\right\}^T, \tag{2.16}$$

where the coefficients $r_{\boldsymbol{k}}^{(j)}$ satisfy

$$\sum_j \left(r_{\boldsymbol{k}}^{(j)}\right)^2 = 1, \quad \sum_j \alpha_{\boldsymbol{k}}^{(j)} r_{\boldsymbol{k}}^{(j)} = 0. \tag{2.17a,b}$$

The last condition, (2.17*b*), is a consequence of the eigenvector equation. In the n-scalar problem, the solution $\omega = 0$ is $(n-1)$-fold degenerate. The $n-1$ eigenvectors $\boldsymbol{v}_{\boldsymbol{k}}^{(0)}$ can be orthonormalized using standard procedures.

The following comments are appropriate.

(i) The eigenvectors $\boldsymbol{v}_{\boldsymbol{k}}^{(\pm)}$ represent waves propagating to the left and right in the x-direction, while the zero-frequency eigenvector $\boldsymbol{v}_{\boldsymbol{k}}^{(0)}$ can be interpreted as a non-oscillatory mode.

(ii) The presence of $a_{\boldsymbol{k}}$ in $\boldsymbol{v}_{\boldsymbol{k}}^{(\pm)}$ breaks the symmetry between left- and right-travelling waves. This is a consequence of the symmetry-breaking term $\lambda_{\boldsymbol{k}}$ appearing in the equations of motion. When $\lambda_{\boldsymbol{k}} \equiv 0$, we find $\cos a_{\boldsymbol{k}} = \sin a_{\boldsymbol{k}} = 1/\sqrt{2}$ and the symmetry is restored.

(iii) The jth scalar field can be decoupled from the linear dynamics by turning off the gradient $\sigma_j \rightarrow 0$, or, equivalently, $\alpha_{\boldsymbol{k}}^{(j)} \rightarrow 0$. Condition (2.17*b*) is then satisfied by $r_{\boldsymbol{k}}^{(j)} = 1$, $r_{\boldsymbol{k}}^{(i)} = 0$ for all $i \neq j$. In particular, setting $\partial_z \langle \chi \rangle \rightarrow 0$ decouples $\tilde{\chi}$, so that the zero-frequency mode becomes simply

$$\boldsymbol{v}_{\boldsymbol{k}}^{(0)} = \left\{0 \quad 0 \quad 1\right\}^T. \tag{2.18}$$

If, in addition, $\lambda_{\boldsymbol{k}} = 0$, the usual Elsasser modes are recovered:

$$\boldsymbol{v}_{\boldsymbol{k}}^{(\pm)} = \frac{1}{\sqrt{2}} \left\{1 \quad \pm 1 \quad 0\right\}^T. \tag{2.19}$$

To illustrate the general formulation, let us consider two simple examples: two-dimensional MHD turbulence in the presence of rotation and stratification.

2.3. *First illustration: β-plane MHD*

We first modify the two-dimensional MHD equations to incorporate the effects of rotation on a spherical shell, i.e.

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\nabla^2\psi + \beta v_z = (B_0\partial_x + \boldsymbol{b} \cdot \nabla)\nabla^2\tilde{A} + \nu\nabla^2\nabla^2\psi, \tag{2.20a}$$

$$(\partial_t + \boldsymbol{v} \cdot \nabla)\tilde{A} + v_z B_0 = \eta\nabla^2\tilde{A}. \tag{2.20b}$$

The (x, z)-plane lies on the surface of a sphere, with x (and the mean poloidal field B_0) pointing to the west, z increasing to the north, and y pointing radially inwards. Here and throughout, β is the latitudinal gradient in the locally vertical component of the planetary vorticity – not, we emphasize, the plasma beta-parameter; nor is it the turbulent resistivity, which we have denoted η_T to avoid confusion.

The most obvious effect of β is to modify the linear theory. Instead of the usual Alfvén waves travelling in each direction along B_0, the linearized dissipationless equations yield the following dispersion relation:

$$\omega^2 + \Omega_{\boldsymbol{k}}^{RW}\omega - \Omega_{\boldsymbol{k}}^{AW^2} = 0, \tag{2.21}$$

where $\Omega_{\boldsymbol{k}}^{AW} = B_0 k_x$ and $\Omega_{\boldsymbol{k}}^{RW} = -\beta k_x/k^2$ are the usual Alfvén and Rossby wave frequencies. Equation (2.21) has the solutions

$$\omega_{\boldsymbol{k}}^{\pm} = \tfrac{1}{2}\Omega_{\boldsymbol{k}}^{RW} \pm \sqrt{\left(\tfrac{1}{2}\Omega_{\boldsymbol{k}}^{RW}\right)^2 + \left(\Omega_{\boldsymbol{k}}^{AW}\right)^2}, \tag{2.22}$$

so that the linear modes represent hybrid Rossby–Alfvén waves propagating with and against the direction of rotation. Note that β breaks the symmetry between pro- and retrograde propagation.

In the limit $\beta \to 0$, or $k \ll \sqrt{\beta/B_0}$, the eigenmodes reduce to the usual Elsasser modes. Conversely, when $B_0 \to 0$, or $k \gg \sqrt{\beta/B_0}$, the velocity and magnetic fluctuations decouple, with eigenvalues of $\Omega_{\boldsymbol{k}}^{RW}$ and 0, respectively. Thus, the finite-frequency mode is a Rossby wave, while the zero-frequency eigenmode recovered in this limit may be interpreted as a spatially extended zonal flow (Rhines 1975; Diamond *et al.* 2007).

Equations (2.20*a*) and (2.20*b*) constitute a simple model of the solar tachocline, a thin ($< 0.04R_{\odot}$) shear layer located between the convection zone and the radiation zone (Spiegel & Zahn 1992; Tobias 2005). Turbulence in the solar tachocline is quasi-geostrophic MHD turbulence confined to a thin spherical shell by strong stratification in the radiation zone, and driven from above by convectively overshooting plumes (Miesch 2005).

2.4. *Second illustration: stratified MHD*

In our second example, two-dimensional MHD in the presence of stable stratification, we begin with the Navier–Stokes equation and the continuity equation for an incompressible hydromagnetic fluid in the presence of an external gravitational field $-g\hat{\boldsymbol{z}}$:

$$\rho\,(\partial_t + \boldsymbol{v}\cdot\nabla)\,\boldsymbol{v} = -\nabla P_{eff} + \frac{\boldsymbol{B}\cdot\nabla\boldsymbol{B}}{4\pi} - \rho g\hat{\boldsymbol{z}} + \rho\nu\nabla^2\boldsymbol{v}, \tag{2.23a}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\rho = \mathscr{D}\nabla^2\rho, \tag{2.23b}$$

where P_{eff} incorporates both magnetic and thermal pressure. This time, we assume that an intense toroidal field confines motion to the (x, z)-plane while a weaker poloidal field lies in the positive x-direction.

In addition to the streamfunction ψ and the magnetic potential $\tilde{A}$, we allow fluctuations $\tilde{\rho}$ in the density; consistent with a Boussinesq approximation, however, we permit such fluctuations to appear only in the buoyancy term in the Navier–Stokes equation. Taking the curl of (2.23*a*), the dissipationless equations of motion become

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\Omega_y = (\boldsymbol{b}\cdot\nabla + B_0\partial_x)j_y - \frac{g}{\langle\rho\rangle}\partial_x\tilde{\rho}, \tag{2.24a}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\tilde{A} = v_z B_0, \tag{2.24b}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\tilde{\rho} = -v_z\frac{\partial\langle\rho\rangle}{\partial z}, \tag{2.24c}$$

where the magnetic field is measured in velocity units.

Equations (2.24a)–(2.24c) can be written in the form (2.3a)–(2.3c) by introducing

$$\chi = \frac{g}{N}\ln\rho, \tag{2.25}$$

where N, the Brunt–Väisälä frequency for stable stratification, is defined by

$$N^2 = -\frac{g}{\langle\rho\rangle}\frac{\partial\langle\rho\rangle}{\partial z} \geqslant 0, \tag{2.26}$$

assumed constant over the vertical distances of interest. Thus we have, for $\tilde{\rho} \ll \langle\rho\rangle$,

$$\chi \approx \frac{g}{N}\left(\ln\langle\rho\rangle + \frac{\tilde{\rho}}{\langle\rho\rangle}\right), \tag{2.27}$$

so that the mean gradient of χ is simply

$$N = -\frac{\partial\langle\chi\rangle}{\partial z} \geqslant 0, \tag{2.28}$$

and the dissipationless equations of motion become

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\Omega_y = (B_0\partial_x + \boldsymbol{b}\cdot\nabla)\nabla^2\tilde{A} - N\partial_x\tilde{\chi}, \tag{2.29a}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\tilde{A} = B_0\partial_x\psi, \tag{2.29b}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\tilde{\chi} = N\partial_x\psi. \tag{2.29c}$$

The linear eigenfrequencies of this system are zero and $\pm\Omega_{\boldsymbol{k}}$, where

$$\Omega_{\boldsymbol{k}} = \sqrt{\left(\Omega_{\boldsymbol{k}}^{AW}\right)^2 + \left(\Omega_{\boldsymbol{k}}^{IGW}\right)^2}. \tag{2.30}$$

Again, $\Omega_{\boldsymbol{k}}^{AW} = B_0k_x$ is the Alfvén wave frequency, while $\Omega_{\boldsymbol{k}}^{IGW} = Nk_x/|\boldsymbol{k}|$ is the frequency of an internal gravity wave. Note that, as $g_2(k) \propto k^{-1}$, the χ-nonlinearity vanishes on the right-hand side of (2.29a).

Note again that the usual Elsasser modes are recovered in the limits $N \to 0$ or $k \ll N/B_0$. However, in contrast to the β-plane MHD model, there always exists a zero-frequency mode in stratified MHD. In three-dimensional stratified neutral fluid dynamics, such a mode is referred to as a 'vortical mode' (see, for instance, Staquet & Sommeria 2002). Such modes cannot exist in two-dimensional stratified flow.

3. Wave-interaction-driven flux in wavy MHD

The theory of weak or wave turbulence has a distinguished pedigree, beginning with the pioneering work by Peierls (1929) on lattice vibrations and of Phillips (1960), Hasselmann (1966) and Benney & Newell (1969) on the interactions of waves in neutral fluids. Early references on wave turbulence in plasmas include Sagdeev & Galeev (1969) and Davidson (1972). Zakharov, L'vov & Falkovich (1992) or Krommes (2002) give a thorough review and a comprehensive list of references.

In nearly all cases, weak turbulence theory has as its goal the calculation of the spectral transfer of energy in k-space, with the aim of determining the fluctuation spectral distribution. By contrast, our aim here is to calculate the spatial transport in x-space induced by wave interactions. Obviously, these two processes are related, yet distinct, in principle.

The lowest-order contribution to the transfer of energy among waves, so far as we will be concerned, comes from the interaction of a triad of waves $(\boldsymbol{k}, \omega)$, $(\boldsymbol{k}', \omega')$ and

$(\boldsymbol{k}'', \omega'')$ satisfying the selection rules

$$\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{0}, \tag{3.1a}$$

$$\omega + \omega' + \omega'' = 0. \tag{3.1b}$$

The transfer of energy during coherent three-wave interactions is analogous to that which occurs in a free asymmetric top, during which the kinetic energy of oscillations about the unstable axis of rotation is slowly transferred to the stable axes. This analogy is incomplete, however: as is well-known, the phase-space orbits of the free asymmetric top are periodic, indicating that energy transfer is reversible. On the other hand, irreversible energy transfer arises for an ensemble of wave resonances. The physical origin of irreversibility in wave turbulence theory may be thought of as chaos induced by the overlap of multiple wave resonances. In practice, stochasticity must be incorporated into the formalism by hand via the random phase approximation (RPA); see Wersinger *et al.* (1980) for a discussion of the dynamical origins of the RPA.

The validity of wave turbulence theory depends crucially upon the relationship between three time scales, which are related to the following.

(i) The mismatch (or lack thereof) in the wave frequencies $\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'} = \omega_{\boldsymbol{k}}^{(\alpha)} + \omega_{\boldsymbol{k}'}^{(\alpha')} + \omega_{\boldsymbol{k}''}^{(\alpha'')}$ for some triad $\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{0}$.

(ii) The dispersion in the frequency mismatch. For a spectrum of width $\Delta\boldsymbol{k}$, centred on $\boldsymbol{k}_0$, we can estimate this as

$$\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'} - \Delta\omega_{\boldsymbol{k},\boldsymbol{k}_0} \approx \frac{\partial \Delta\omega_{\boldsymbol{k},\boldsymbol{k}'}}{\partial \boldsymbol{k}'} \cdot \Delta\boldsymbol{k}. \tag{3.2}$$

The inverse of this quantity is the time during which a given wave triad remains sufficiently coherent to interact, τ_{int}.

(iii) The energy transfer time, being the slow time scale on which interacting waves transfer energy. This can be estimated from the wave-kinetic equation (Zakharov *et al.* 1992) as the inverse of the energy transfer rate

$$\gamma_{NL_{\boldsymbol{k}}} \sim \sum_{\boldsymbol{k}'+\boldsymbol{k}''=-\boldsymbol{k}} \mathscr{M}^2 \left|x_{\boldsymbol{k}'}\right|^2 \delta(\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'}). \tag{3.3}$$

Here, $\mathscr{M}$ refers to the elements of the interaction matrix.

In a renormalized theory, the delta function appearing in (3.3) is broadened by γ_{NL}. A necessary condition for the neglect of resonance broadening is

$$\gamma_{NL} \ll \left| \frac{\partial \Delta\omega_{\boldsymbol{k},\boldsymbol{k}'}}{\partial \boldsymbol{k}'} \cdot \Delta\boldsymbol{k} \right|. \tag{3.4a}$$

Thus, wave turbulence theory describes the transfer of energy among a triad of waves satisfying $\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'}$ when the nonlinear transfer time $\tau_{tr} = \gamma_{NL}^{-1}$ is small compared with the triad interaction time:

$$\tau_{int} \ll \tau_{tr}. \tag{3.4b}$$

Additionally, we demand that the viscous damping rate $\gamma_{\boldsymbol{k}}$ is much smaller than $1/\tau_{ac}$, where τ_{ac} is the spectral auto-correlation time:

$$\gamma_{\boldsymbol{k}} \ll \frac{1}{\tau_{ac}}. \tag{3.5}$$

Note that, for non-dispersive waves, $\Delta\omega_{\boldsymbol{k},\boldsymbol{k}'}$ vanishes for all triads $\boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{0}$, so that wave turbulence theory is inapplicable in this case. However, dispersion alone is not sufficient: a sufficiently broad spectrum of waves is required so that $\tau_{int} \ll \tau_{tr}$.

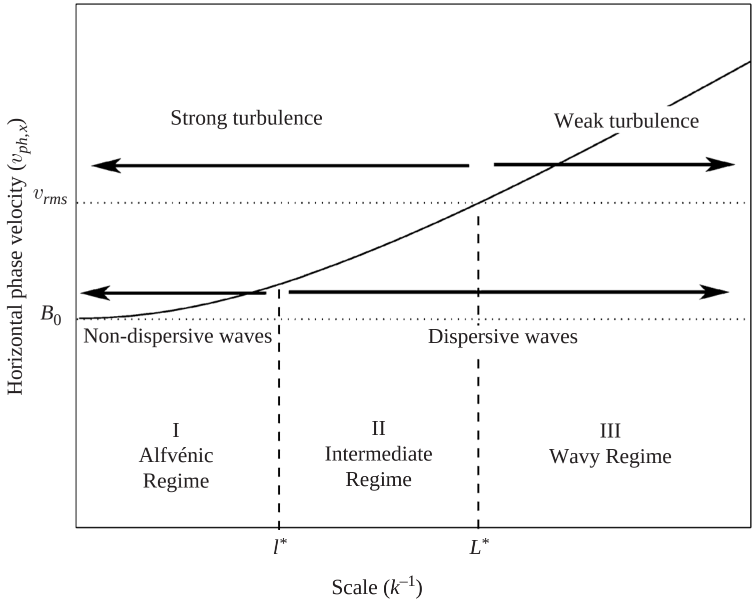

FIGURE 1. Spectral ranges of wavy MHD in two dimensions.

Finally, γ_{NL} characterizes the strength of the interaction, and will be small if the wave amplitude or the interaction coefficient is small. Thus, condition (3.4*a*), and hence the validity of wave turbulence theory, requires a broad spectrum of weakly interacting dispersive waves.

3.1. *Spectral ranges of wavy MHD*

Two lengths scales arise naturally in wavy MHD turbulence. A linear length scale ℓ^* separates those scales on which the eigenmodes are non-dispersive (Alfvén waves) from those scales with dispersive eigenmodes. An additional cross-over length scale L^* divides the spectrum into weakly and strongly turbulent regimes. These scales divide wavenumber space into three ranges: Alfvénic, intermediate and wavy. It is the wavy range for which we can apply wave turbulence theory and identify τ_c as the triad coherence time, and thus will be the primary focus of the remainder of the paper.

Let us assume that the horizontal phase velocity of the modified eigenmodes, $v_{ph,x}(k) = |\omega/k_x|$, is a monotonically decreasing function of k – i.e. monotonically increasing with scale $\ell = k^{-1}$ (figure 1). Note that this is the case for both β-plane MHD and stratified MHD. For small scales, the phase velocity will asymptotically approach the constant Alfvén speed B_0, whereas at large scales, the phase velocity is a function of ℓ, indicating dispersion.

This observation motivates us to define the characteristic linear length scale ℓ^* as that scale for which

$$v_{ph,x}(k) \approx B_0, \quad \forall k > \ell^{*-1}. \tag{3.6}$$

Expressions for ℓ^* for β-plane MHD and stratified MHD are given in table 1. Notice that, on scales $\ell \ll \ell^*$, the linear eigenmodes are approximately Alfvénic, while on scales $\ell \gg \ell^*$, the eigenmodes behave more like Rossby waves (in the case of β-plane MHD) and internal gravity waves (in the case of stratified MHD).

Consideration of the turbulent decorrelation of waves introduces a second natural length scale to wavy MHD turbulence. At each wavenumber $\boldsymbol{k}$, there is associated

	β-plane MHD	Stratified MHD	Wavy MHD
ℓ^*	$\sqrt{B_0/\beta}$	B_0/N	$v_{ph,x}(\ell^{*-1}) \approx B_0$
$\lambda_{\boldsymbol{k}}$	$-\beta k_x/\lvert\boldsymbol{k}\rvert^2$	0	$f(\boldsymbol{k})\,k_x$
$\Omega_{1,\boldsymbol{k}}$	$B_0 k_x$	$B_0 k_x$	$\sigma_1 g_1(\boldsymbol{k})\,k_x$
$\Omega_{2,\boldsymbol{k}}$	0	$N k_x/\lvert\boldsymbol{k}\rvert$	$\sigma_2 g_2(\boldsymbol{k})\,k_x$
$\ell \gg \ell^*$	Rossby waves	Internal waves	Dispersive waves
$\ell \ll \ell^*$	Alfvén waves	Alfvén waves	Alfvén waves

TABLE 1. Summary of linear theory for β-plane, stratified, and wavy MHD.

	β-plane MHD	Stratified MHD	Wavy MHD
L^*	$\sqrt{\tilde{V}/\beta}$	$\tilde{V}/N$	$v_{ph,x}(L^{*-1}) \approx \tilde{V}$
$\ell \gg L^*$	Rossby wave interactions	Internal wave interactions	Dispersive wave interactions
$\ell \ll L^*$	Forward cascade	Forward cascade	Forward cascade

TABLE 2. Turbulent ranges and energy transfer for β-plane stratified, and wavy MHD.

with that wavenumber a frequency $\omega_{\boldsymbol{k}}$ and a decorrelation rate $d_{\boldsymbol{k}}$, describing the rate at which turbulent broadening washes out wave interactions. When the wave mismatch $\Delta\omega$ is much smaller than the decorrelation rate $d_{\boldsymbol{k}}$, waves are washed out before they can interact resonantly. Therefore, the cross-over length scale L^*, defined as the value of $|\boldsymbol{k}|^{-1}$ for which $\Delta\omega = d_{\boldsymbol{k}}$, marks the transition from eddy-dominated turbulence on small scales to wave-dominated turbulence on large scales.

We can estimate L^* by assuming $d_{\boldsymbol{k}} \approx k\tilde{V}$ and $\Delta\omega \approx \omega_{\boldsymbol{k}}$. Here, $\tilde{V}$ is a typical eddy velocity, so L^* will exhibit some sensitivity to the spectrum. The scale L^* is then defined by

$$v_{ph,x}(L^{*-1}) \approx \tilde{V}. \tag{3.7}$$

The wave-slope

$$\epsilon_{\boldsymbol{k}} = \frac{d_{\boldsymbol{k}}}{\omega_{\boldsymbol{k}}} \approx \frac{\tilde{V}}{v_{ph}(\boldsymbol{k})}, \tag{3.8}$$

which is greater than unity for scales $\ell < L^*$ and less than unity for scales $\ell > L^*$, plays the role of an expansion parameter. On scales $\ell > L^*$, the equations of motion can be expanded in powers of ϵ or, equivalently, the fluctuation amplitude. In this regime, the turbulence is weak and energy transfer is via nonlinear wave interactions. On the other hand, on scales $\ell < L^*$, the turbulence is strong, and energy transfer is via the usual forward two-dimensional MHD cascade.

Table 2 gives expressions for L^* for the cases of β-plane MHD and stratified MHD. Note that, in β-plane MHD, L^* is a generalization of the Rhines scale of quasi-geostrophic turbulence (Rhines 1975; Diamond *et al.* 2007); for stratified MHD it is a generalization of the Ozmidov scale of oceanic internal-wave turbulence (Ozmidov 1965). The solutions to (3.6) and (3.7) are shown in figure 1. The ratio L^*/ℓ^* is an increasing function of the magnetic Mach number $M = \tilde{V}/B_0$, so that we can expect $L^* \gg \ell^*$ in general. Thus, wavenumber space is divided into three ranges. Starting with the smallest scales, these are as follows.

(I) The 'Alfvén range', $\ell < \ell^*, L^*$: the modes in this range are strongly interacting eddies and Alfvén waves. The dynamics is that of ordinary two-dimensional MHD, so that the only source of irreversibility is molecular diffusion (i.e. resistivity, viscosity), and the contribution to the flux of magnetic potential from these scales is predicted to be quenched according to (1.2). Thus, the turbulent resistivity associated with this range will be a function of Rm:

$$\eta_T^{(\mathrm{I})} = \eta_T^{(\mathrm{I})}(Rm^{-1}).$$

(II) The 'intermediate range', $\ell^* < \ell < L^*$: waves in this range are dispersive, but are washed out by turbulent broadening before they can interact. A quasi-linear closure of the flux of magnetic potential can be obtained by making a straightforward, if tedious, calculation analogous to that of § 1. However, as the only source of irreversibility is again molecular diffusion, we expect that the turbulent resistivity associated with these scales will again be quenched:

$$\eta_T^{(\mathrm{II})} = \eta_T^{(\mathrm{II})}(Rm^{-1}).$$

(III) The 'wavy range', $\ell > \ell^*, L^*$: here, waves are dispersive and weakly interacting. Wave resonances give rise to irreversibility which is not tied to the molecular resistivity. Weak turbulence theory is applicable, and the turbulent resistivity can be calculated via an expansion in powers of the turbulent amplitude:

$$\eta_T^{(\mathrm{III})} = \overset{(2)}{\eta_T} + \overset{(4)}{\eta_T} + \overset{(6)}{\eta_T} + \cdots .$$

As we shall see, the second-order contribution to $\eta_T^{(\mathrm{III})}$ is directly tied to the collisional resistivity; however, the contributions at fourth order and higher are not. Therefore, we can express the overall turbulent resistivity as

$$\eta_T = \eta_T^{\mathrm{QL}}(Rm^{-1}, \ldots) + \overset{(4)}{\eta_T}(Rm^0) + \overset{(6)}{\eta_T}(Rm^0) + \cdots . \tag{3.9}$$

Thus we see that, in the absence of weakly interacting dispersive waves, the turbulent resistivity is quenched. When the linear eigenmodes are dispersive in some range of wavenumber space, possible Rm-dependent quenching of η_T can be circumvented by the fourth-order contribution to η_T. Because the contributions to $\eta_T^{(\mathrm{III})}$ at fourth order are independent of Rm, it may be the case that the fourth-order contribution, arising from nonlinear wave interactions, exceeds the quenched contribution, which scales as Rm^{-1}. Note that this last may have additional dependencies upon the Reynolds number Re, the magnetic Prandtl number Pm, and so on, as indicated by the ellipses in the argument of η_T^{QL}.

3.2. *Calculation of the wave-interaction-driven flux*

We now calculate the vertical flux of a scalar field θ_j due to nonlinear wave interactions

$$\begin{aligned}\Gamma_j &= \langle \theta_j \delta v_z \rangle + \langle v_x \delta\theta_j \rangle \\ &= \mathrm{Re} \sum_{\boldsymbol{k},\omega} \mathrm{i}k_x (\theta^*_{j,\boldsymbol{k}\omega} \delta\psi_{\boldsymbol{k}\omega} - \psi^*_{\boldsymbol{k}\omega} \delta\theta_{j,\boldsymbol{k}\omega}),\end{aligned} \tag{3.10}$$

where all summations are implicitly over scales $|\boldsymbol{k}|^{-1} \geqslant L^*$. The responses of the fluid and the scalar fields to wave interactions can be expanded in powers of the wave amplitude:

$$\delta x_{\boldsymbol{k}\omega} = \overset{(1)}{\delta x}_{\boldsymbol{k}\omega} + \overset{(2)}{\delta x}_{\boldsymbol{k}\omega} + \overset{(3)}{\delta x}_{\boldsymbol{k}\omega} \ldots . \tag{3.11}$$

The linear response in the wavy range $|\boldsymbol{k}|^{-1} \geqslant L^*$ is simply due to wave oscillations, so that

$$\overset{(1)}{\delta x}_{\boldsymbol{k}\omega} = x_{\boldsymbol{k}\omega}. \tag{3.12}$$

We can express the wave amplitudes in terms of the wave displacement in the z-direction:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} = v_z, \tag{3.13}$$

so that

$$\psi_{\boldsymbol{k}\omega} = -\frac{\omega}{k_x}\varepsilon_{\boldsymbol{k}\omega}. \tag{3.14}$$

Similarly, the linearized advection equation for the C_j gives

$$C_{j,\boldsymbol{k}\omega} = \frac{\Omega_{\boldsymbol{k}}^{(j)}}{k_x}\varepsilon_{\boldsymbol{k}\omega}. \tag{3.15}$$

The higher-order responses obey the equations of motion

$$-\mathrm{i}\omega\delta x^a_{\boldsymbol{k}\omega} - \mathrm{i}\mathscr{L}^{ab}_{\boldsymbol{k}}\delta x^b_{\boldsymbol{k}\omega} = N^a_{\boldsymbol{k}\omega}(x,\delta x). \tag{3.16}$$

This has the formal solution

$$\delta x^a_{\boldsymbol{k}\omega} = \tau^{ab}_{\boldsymbol{k}\omega}N^b_{\boldsymbol{k}\omega}(x,\delta x). \tag{3.17}$$

The Green function associated with (3.16) is

$$\tau^{ab}_{\boldsymbol{k}\omega} = T^{-1a\alpha}_{\boldsymbol{k}\omega}\frac{\mathrm{i}}{\omega+\omega^{(\alpha)}_{\boldsymbol{k}}+\mathrm{i}\gamma^{(\alpha)}_{\boldsymbol{k}\omega}}T^{\alpha b}_{\boldsymbol{k}}, \tag{3.18}$$

where $\boldsymbol{T}^{-1}_{\boldsymbol{k}} = \boldsymbol{T}^{\boldsymbol{T}}_{\boldsymbol{k}}$ is the matrix of eigenvectors of $\mathscr{L}_{\boldsymbol{k}}$. The energy transfer rate $\gamma^{(\alpha)}_{\boldsymbol{k}\omega}$ for the mode α, calculated from the wave-kinetic equation (Zakharov *et al.* 1992), is of order $\epsilon^2_{\boldsymbol{k}}$, so that we can expand $\tau_{\boldsymbol{k}\omega}$ as

$$\begin{aligned}\tau^{ab}_{\boldsymbol{k}\omega} &= \overset{(0)}{\tau}{}^{ab}_{\boldsymbol{k}\omega} + \overset{(2)}{\tau}{}^{ab}_{\boldsymbol{k}\omega} + \overset{(4)}{\tau}{}^{ab}_{\boldsymbol{k}\omega} + \cdots \\ &= T^{-1a\alpha}_{\boldsymbol{k}\omega}\frac{\mathrm{i}}{\omega+\omega^{(\alpha)}_{\boldsymbol{k}}+\mathrm{i}0^+}\left\{1+\left(\frac{-\mathrm{i}\gamma^{(\alpha)}_{\boldsymbol{k}\omega}}{\omega+\omega^{(\alpha)}_{\boldsymbol{k}}}\right)+\left(\frac{-\mathrm{i}\gamma^{(\alpha)}_{\boldsymbol{k}\omega}}{\omega+\omega^{(\alpha)}_{\boldsymbol{k}}}\right)^2+\cdots\right\}T^{\alpha b}_{\boldsymbol{k}},\end{aligned} \tag{3.19}$$

where the presence of 0^+ ensures causality.

Equation (3.17) can then be solved iteratively:

$$\begin{aligned}\overset{(2)}{\delta x}{}^a_{\boldsymbol{k}\omega} &= \overset{(0)}{\tau}{}^{ab}_{\boldsymbol{k}\omega}\,N^b_{\boldsymbol{k}\omega}(x,x),\\ \overset{(3)}{\delta x}{}^a_{\boldsymbol{k}\omega} &= \overset{(0)}{\tau}{}^{ab}_{\boldsymbol{k}\omega}\,N^b_{\boldsymbol{k}\omega}(x,\overset{(2)}{\delta x}),\\ \overset{(4)}{\delta x}{}^a_{\boldsymbol{k}\omega} &= \overset{(3)}{\delta x}{}^a_{\boldsymbol{k}\omega}\overset{(2)}{\tau}{}^{ab}_{\boldsymbol{k}\omega}\,N^b_{\boldsymbol{k}\omega}(x,x)+\overset{(0)}{\tau}{}^{ab}_{\boldsymbol{k}\omega}\,N^b_{\boldsymbol{k}\omega}(x,\overset{(3)}{\delta x}),\\ &\vdots\quad\vdots\end{aligned}$$

The flux of the scalar θ_j is then

$$\Gamma_j = \overset{(2)}{\Gamma}_j + \overset{(4)}{\Gamma}_j + \overset{(6)}{\Gamma}_j + \cdots, \tag{3.20}$$

where, using (3.14) and (3.15),

$$\overset{(n)}{\Gamma}_j = \mathrm{Re}\sum_{\boldsymbol{k},\omega}\frac{\mathrm{i}}{g_j}\left\{\Omega^{(j)}_{\boldsymbol{k}}\,\overset{(n-1)}{\delta\psi}_{\boldsymbol{k}\omega}+\omega\,\overset{(n-1)}{\delta C}_{j,\boldsymbol{k}\omega}\right\}\varepsilon^*_{\boldsymbol{k}\omega} \qquad (n=2,4,6,\ldots) \tag{3.21}$$

3.3. *Calculation of the flux to second order*

When ω is real, (3.21) vanishes identically for the case of $n=2$, as can be seen by substituting (3.14) and (3.15) into (3.21). Hence, wave interactions make no second-order contribution to the flux of the scalar field θ_j. Rather, the flux to second order is tied to the collisional diffusivities, as can be seen by a direct calculation:

$$\overset{(2)}{\Gamma_j} = -\mathrm{Re} \sum_{\boldsymbol{k},\omega} \mathrm{i} k_x \theta_{j,\boldsymbol{k}\omega} \psi^*_{\boldsymbol{k}\omega}. \tag{3.22}$$

The linearized equation of motion for the scalar field θ_j is

$$(-\mathrm{i}\omega + \mathscr{D}_j k^2)\theta_{j,\boldsymbol{k}\omega} = \mathrm{i}\sigma_j k_x \psi_{\boldsymbol{k}\omega}, \tag{3.23}$$

where $\mathscr{D}_j$ is the collisional diffusivity associated with θ_j. The wave-frequency ω will, in general, possess a real and imaginary part $\omega = \omega_{\boldsymbol{k}} + \mathrm{i}\gamma_{\boldsymbol{k}}$ where $\gamma_{\boldsymbol{k}}$ is the linear growth rate (negative for damping). The growth rate will, in general, be a function of ν, η_c and $\mathscr{D}$, as well as $\boldsymbol{k}$.

Substituting (3.23) into (3.22) and taking only the real part yields the result

$$\overset{(2)}{\Gamma_j} = -\frac{\partial\langle\theta_j\rangle}{\partial z} \sum_{\boldsymbol{k},\omega} \frac{|\gamma_{\boldsymbol{k}} + \mathscr{D}_j k^2|}{\omega_{\boldsymbol{k}}^2} \langle v^2\rangle_{\boldsymbol{k}\omega}, \tag{3.24}$$

where we have assumed $|\omega_{\boldsymbol{k}}| \gg \gamma_{\boldsymbol{k}}, \mathscr{D}_j k^2$.

We show in Appendix B that, for a spectrum of Alfvén waves, the growth rate is

$$\gamma_{\boldsymbol{k}} = -\tfrac{1}{2}(\eta_c + \nu)\, k^2. \tag{3.25}$$

Thus, the turbulent resistivity to second order will be

$$\overset{(2)}{\eta_T} \approx \sum_{\boldsymbol{k}} \frac{|\eta_c - \nu|\, k^2}{2\omega_{\boldsymbol{k}}^2} \langle v^2\rangle_{\boldsymbol{k}\omega}, \tag{3.26}$$

where $\omega_{\boldsymbol{k}} = \Omega_{\boldsymbol{k}}^{\mathrm{AW}}$ is the frequency of an Alfvén wave. Note that (3.26) vanishes for unity magnetic Prandtl number.

In addition to a spectrum of Alfvén waves, expressions for γ for a spectrum of Rossby–Alfvén waves and magneto–internal waves are derived in Appendix B, along with the corresponding turbulent resistivities. The general result, however, is the same: the second-order flux in the presence of a spectrum of waves, like the quasi-linear flux discussed for MHD without waves in § 1, is tied to the diffusivities ν, η_c and $\mathscr{D}$, and so is quenched.

3.4. *Calculation of the flux to fourth order*

Hereinafter, we use the shorthand $\psi_{\boldsymbol{k}\omega} \rightarrow \psi$, $\psi_{\boldsymbol{k}'\omega'} \rightarrow \psi'$, etc. The fourth-order contribution to the flux is then

$$\overset{(4)}{\Gamma_j} = \mathrm{Re} \sum_{\boldsymbol{k},\omega} \frac{\mathrm{i}}{g_j} \left[\Omega_{\boldsymbol{k}}^{(j)}\, \delta\overset{(3)}{\psi} + \omega\, \delta\overset{(3)}{C_j} \right] \varepsilon^*. \tag{3.27}$$

The third-order response obeys

$$\delta\overset{(3)}{x^a} = \overset{(0)}{\boldsymbol{\tau}^{ab}_{\boldsymbol{k}\omega}}\, N^b(x'^*, \delta\overset{(2)}{x''^*}), \tag{3.28}$$

so that the expression in square brackets in (3.27) becomes

$$\left[\Omega_{\boldsymbol{k}}^{(j)} \overset{(3)}{\delta\psi} + \omega \overset{(3)}{\delta C_j}\right] = \left(\Omega_{\boldsymbol{k}}^{(j)} T_{\boldsymbol{k}}^{-1\psi\alpha} + \omega T_{\boldsymbol{k}}^{-1j\alpha}\right) \frac{\mathrm{i}}{\omega + \omega_{\boldsymbol{k}}^{(\alpha)} + \mathrm{i}0^+} T_{\boldsymbol{k}}^{\alpha b} N^b(x'^*, \overset{(2)}{\delta x''^*}). \quad (3.29)$$

The elements of the unitary matrix of eigenvectors $\boldsymbol{T}^{-1} = \boldsymbol{T}^{\mathrm{T}}$ obey

$$\Omega_{\boldsymbol{k}}^{(j)} T_{\boldsymbol{k}}^{-1\psi\alpha} = \omega_{\boldsymbol{k}}^{(\alpha)} T_{\boldsymbol{k}}^{-1j\alpha}. \quad (3.30)$$

Thus, the fourth-order flux reduces to

$$\overset{(4)}{\Gamma^{(j)}} = -\mathrm{Re} \sum_{\boldsymbol{k},\omega} \frac{1}{g_j} N^{(j)}(x'^*, \overset{(2)}{\delta x''^*}) \varepsilon^*. \quad (3.31)$$

The nonlinearity $N^{(j)}$ is obtained from (2.9) with first- and second-order fields:

$$N^{(j)}(x'^*, \overset{(2)}{\delta x''^*}) = \tfrac{1}{2} \sum_{\substack{\boldsymbol{k}'+\boldsymbol{k}''=-\boldsymbol{k} \\ \omega'+\omega''=-\omega}} \left(\boldsymbol{k}\cdot\boldsymbol{k}' \times \hat{y}\right) \left(\frac{g_j}{g_j'} C_j'^* \overset{(2)}{\delta\psi''^*} - \frac{g_j}{g_j''} \psi'^* \overset{(2)}{\delta C_j''^*} \right). \quad (3.32)$$

Substituting (3.32) into (3.31) yields

$$\overset{(4)}{\Gamma_j} = \tfrac{1}{2} \mathrm{Re} \sum_{\substack{\delta\boldsymbol{k}=\boldsymbol{0} \\ \delta\omega=0}} (\boldsymbol{k}\cdot\boldsymbol{k}' \times \hat{y} \left(\sigma_j \overset{(2)}{\delta\psi''^*} + \frac{\omega}{k_x} \frac{\overset{(2)}{\delta C_j''^*}}{g_j''} \right) \varepsilon^* \varepsilon'^*, \quad (3.33)$$

where the summation is over triads $\{(\boldsymbol{k},\omega), (\boldsymbol{k}',\omega'), (\boldsymbol{k}'',\omega'')\}$ such that

$$\delta\boldsymbol{k} = \boldsymbol{k} + \boldsymbol{k}' + \boldsymbol{k}'' = 0, \quad (3.34a)$$

$$\delta\omega = \omega + \omega' + \omega'' = 0. \quad (3.34b)$$

The first term in the brackets in (3.33) vanishes upon symmetrizing with respect to exchange of $(\boldsymbol{k},\omega)$ and $(\boldsymbol{k}',\omega')$. The remaining expression is manifestly not antisymmetric:

$$\overset{(4)}{\Gamma_j} = -\tfrac{1}{4} \mathrm{Re} \sum_{\substack{\delta\boldsymbol{k}=\boldsymbol{0} \\ \delta\omega=0}} (\boldsymbol{k}\cdot\boldsymbol{k}' \times \hat{y}) \left(\frac{\omega}{k_x} - \frac{\omega'}{k_x'} \right) \frac{\overset{(2)}{\delta C_j''^*}}{g_j''} \varepsilon^* \varepsilon'^*. \quad (3.35)$$

The second-order equations of motion can be conveniently obtained from (3.28) by cyclicly permuting the modes $(\boldsymbol{k},\omega)$, $(\boldsymbol{k}',\omega')$ and $(\boldsymbol{k}'',\omega'')$ while preserving conditions (3.34a) and (3.34b):

$$\overset{(2)}{\delta x^{a''*}} = \overset{(0)}{\boldsymbol{\tau}^{ab*}_{\boldsymbol{k}''\omega''}} N^b(x, x'). \quad (3.36)$$

In the spirit of the random phase approximation, we assume that the beat mode $(\boldsymbol{k}'',\omega'')$ is driven directly by the beating of the test mode $(\boldsymbol{k},\omega)$ with the background fluctuation $(\boldsymbol{k}',\omega')$:

$$N^b(x, x') = \mathscr{M}^{bmn}_{\boldsymbol{k}'',\boldsymbol{k},\boldsymbol{k}'} x^m x'^n. \quad (3.37)$$

Using (2.9), (2.10a) and (2.10b) for the elements of the interaction matrix $\mathscr{M}_{\boldsymbol{kpq}}$ we obtain

$$N^{(\psi)}(x, x') = (\boldsymbol{k}\cdot\boldsymbol{k}' \times \hat{\boldsymbol{y}}) n_{\substack{\boldsymbol{k},\boldsymbol{k}' \\ \omega,\omega'}} \varepsilon\varepsilon', \quad (3.38a)$$

$$N^{(j)}(x, x') = (\boldsymbol{k}\cdot\boldsymbol{k}' \times \hat{\boldsymbol{y}}) m^{(j)}_{\substack{\boldsymbol{k},\boldsymbol{k}' \\ \omega,\omega'}} \varepsilon\varepsilon', \quad (3.38b)$$

where

$$n_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = \frac{\omega}{k_x}\frac{\omega'}{k'_x}\frac{k^2-k'^2}{k''^2} - \sum_{\lambda\neq\psi}\sigma_\lambda^2\left\{\frac{k^2g_\lambda^2-k'^2g_\lambda'^2}{k''^2}\right\}, \tag{3.39a}$$

$$m^{(j)}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = -\alpha''_j\frac{\Omega_{\boldsymbol{k}''}}{k''_x}\left(\frac{\omega}{k_x}-\frac{\omega'}{k_x}\right) \equiv -\alpha''_j m_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}. \tag{3.39b}$$

The Green functions for the beat modes contain resonant terms of the form $\mathrm{i}(\omega+\omega'+\omega^{(\pm)}_{\boldsymbol{k}+\boldsymbol{k}'}+\mathrm{i}0^+)^{-1}$ and $\mathrm{i}(\omega+\omega'+\mathrm{i}0^+)^{-1}$. Arranging the second-order equations by their dependence upon these resonances, we find

$$\overset{(2)}{\delta x}{}^{a''*} = (\boldsymbol{k}\cdot\boldsymbol{k}'\times\hat{y})T^{-1a\alpha}_{\boldsymbol{k}''}\frac{\mathrm{i}}{\omega+\omega'+\omega^{(\alpha)}_{\boldsymbol{k}+\boldsymbol{k}'}+\mathrm{i}0^+}P^{\alpha}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\varepsilon\varepsilon', \tag{3.40}$$

where the coefficients $P^{(\alpha)}$ are

$$P^{\alpha}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = T^{\alpha\psi}_{\boldsymbol{k}''}n_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} + \sum_{\lambda\neq\psi}T^{\alpha\lambda}_{\boldsymbol{k}''}m^{(\lambda)}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}. \tag{3.41}$$

Using the properties of the unitary matrix $\boldsymbol{T}$, the coefficients for $\alpha\in\{\pm,0\}$ become simply

$$P^{+}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = n_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\cos a_{\boldsymbol{k}''} - m_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\sin a_{\boldsymbol{k}''}, \tag{3.42a}$$

$$P^{-}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = n_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\sin a_{\boldsymbol{k}''} + m_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\cos a_{\boldsymbol{k}''}, \tag{3.42b}$$

$$P^{0}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = 0. \tag{3.42c}$$

Note that the coefficient multiplying $\mathrm{i}(\omega+\omega'+\mathrm{i}0^+)$ vanishes identically, explicitly eliminating the possibility of two-wave resonances.

Combining our results, we find

$$\overset{(4)}{\Gamma_j} = -\frac{\pi}{8}\frac{\partial\langle\theta_j\rangle}{\partial z}\sum_{\substack{\delta\boldsymbol{k}=\boldsymbol{0}\\ \delta\omega=0}}(\boldsymbol{k}\cdot\boldsymbol{k}'\times\hat{y})^2\sum_{\pm}\mathscr{C}^{\pm}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\delta(\omega+\omega'+\omega^{(\pm)}_{\boldsymbol{k}+\boldsymbol{k}'})|\varepsilon_{\boldsymbol{k}\omega}|^2|\varepsilon_{\boldsymbol{k}'\omega'}|^2, \tag{3.43}$$

where the coupling coefficients $\mathscr{C}^{\pm}$ are given by

$$\mathscr{C}^{\pm}_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = U_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}(1\mp\cos 2a_{\boldsymbol{k}''}) \mp V_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}}\sin 2a_{\boldsymbol{k}''}, \tag{3.44}$$

with

$$U_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = \left(\frac{\omega}{k_x}-\frac{\omega'}{k'_x}\right)^2, \tag{3.45a}$$

$$V_{\substack{\boldsymbol{k},\boldsymbol{k}'\\ \omega,\omega'}} = \frac{k''_x}{\Omega_{\boldsymbol{k}''}}\left(\frac{\omega}{k_x}-\frac{\omega'}{k'_x}\right)\left\{\frac{k^2-k'^2}{k''^2}\frac{\omega}{k_x}\frac{\omega'}{k'_x} - \sum_{\lambda\neq\psi}\sigma_\lambda^2\frac{k^2g_\lambda^2-k'^2g_\lambda'^2}{k''^2}\right\}. \tag{3.45b}$$

We make the following observations about this result.

(i) The fourth-order flux is manifestly symmetric under the exchange of modes $(\boldsymbol{k},\omega)$ and $(\boldsymbol{k}',\omega')$, and hence will not vanish when integrated over resonant triads.

(ii) The turbulent flux of the jth scalar field scales directly with the gradient $\partial_z\langle\theta_j\rangle$. Thus, the fourth-order turbulent diffusivity, defined in the ideal limit by

$$\overset{(4)}{\Gamma_j} = -\frac{\partial\langle\theta_j\rangle}{\partial z}\overset{(4)}{\eta_T},$$

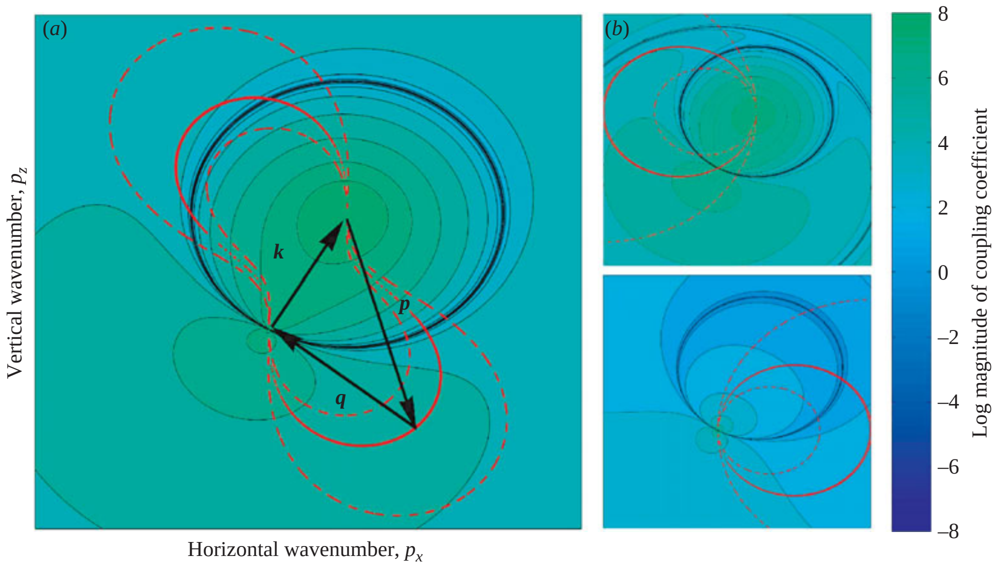

FIGURE 2. Contour plot of the log magnitude of the coupling coefficient for the case of β-plane MHD (for representative wavevector $k = 0.1\,\ell^{*-1}$, $\theta_k = \pi/6$) for triads (a) $\omega_{\boldsymbol{k}}^{(-)} + \omega_{\boldsymbol{p}}^{(-)} + \omega_{\boldsymbol{q}}^{(-)} = 0$, ($b$) $\omega_{\boldsymbol{k}}^{(-)} + \omega_{\boldsymbol{p}}^{(-)} + \omega_{\boldsymbol{q}}^{(+)} = 0$, ($c$) $\omega_{\boldsymbol{k}}^{(-)} + \omega_{\boldsymbol{p}}^{(+)} + \omega_{\boldsymbol{q}}^{(-)} = 0$. Superimposed in red are the corresponding resonance manifolds: the solid curve indicates a zero frequency mismatch while the dashed curve represents a mismatch frequency of 0.5 $\omega_{\boldsymbol{k}}$.

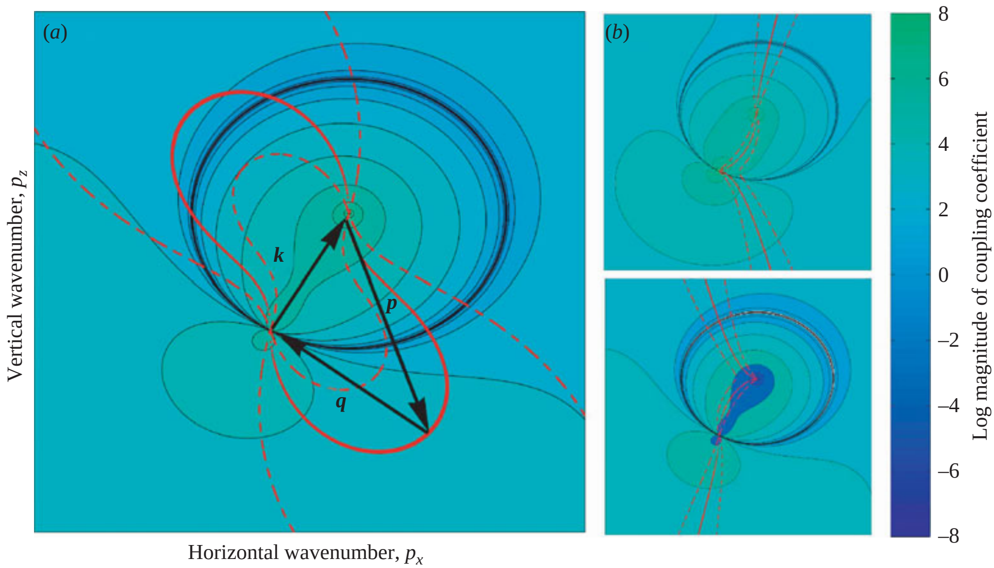

FIGURE 3. Contour plot of the log magnitude of the coupling coefficient for the case of stratified MHD (for representative wavevector $k = 0.1\,\ell^{*-1}$, $\theta_k = \pi/6$) for triads (a) $\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}} = 0$, ($b$) $\omega_{\boldsymbol{k}} - \omega_{\boldsymbol{p}} + \omega_{\boldsymbol{q}} = 0$, ($c$) $\omega_{\boldsymbol{k}} + \omega_{\boldsymbol{p}} - \omega_{\boldsymbol{q}} = 0$. Superimposed in red are the corresponding resonance manifolds: the solid curve indicates a zero frequency mismatch while the dashed curve represents a mismatch frequency of 0.5 $\omega_{\boldsymbol{k}}$.

is identical for all advected scalars. This is perhaps not surprising when we consider that everything in the expansion has been built from the linear theory, which treats each scalar field on an equal footing.

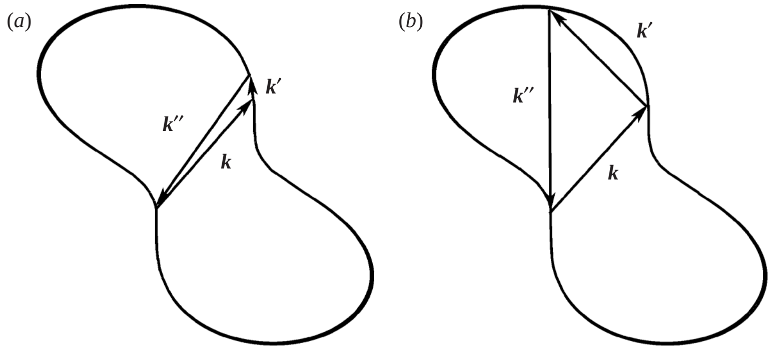

FIGURE 4. (*a*) 'Induced diffusion' and (*b*) 'elastic scattering' triad classes.

(iii) The triad resonance functions $\delta(\omega + \omega' + \omega^{(\pm)}_{\boldsymbol{k}+\boldsymbol{k}'})$ appear as a consequence of retaining the real part of the resonant denominators $\mathrm{i}(\omega + \omega' + \omega^{(\pm)}_{\boldsymbol{k}+\boldsymbol{k}'} + \mathrm{i}0^+)$.

(iv) The coupling coefficients $\mathscr{C}^{\pm}$ are sign-altering when $\lambda_{\boldsymbol{k}} \neq 0$. Hence, there may be some resonant wave triads which give rise to up-gradient transport of magnetic potential. The total flux is therefore a competition between up-gradient and down-gradient contributions.

In figures 2 and 3, the coupling coefficient is evaluated for the cases of β-plane MHD and stratified MHD, respectively, for the representative wavevector $k = 0.1\ \ell^{*-1}$, $\theta_k = \pi/6$, and a range of values of $\boldsymbol{p}$, along with the resonance manifold over which the coupling coefficient is integrated:

$$\omega^{(\alpha)}_{\boldsymbol{k}} + \omega^{(\alpha')}_{\boldsymbol{p}} + \omega^{(\alpha'')}_{\boldsymbol{q}} = 0. \tag{3.46}$$

In each figure, wavevector $\boldsymbol{k}$ is as shown, while $\boldsymbol{p}$ and $\boldsymbol{q}$ can range over the resonance manifold (the solid red curve) such that the wavevectors form the sides of a triangle:

$$\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}. \tag{3.47}$$

If the delta function $\delta(\omega + \omega' + \omega^{(\pm)}_{\boldsymbol{k}+\boldsymbol{k}'})$ is replaced with a broadened resonance, e.g. a Gaussian, then slightly off-resonance triads may also contribute to the fourth-order flux. In figures 2 and 3, the manifold of triads with a frequency mismatch of 0.5 $\omega_{\boldsymbol{k}}$ is outlined by the dashed red curve. We note that, although the contribution to the flux from these off-resonance triads decreases exponentially with increasing frequency mismatch, they may still give rise to an appreciable contribution when integrated over all of $\boldsymbol{k}$ and $\boldsymbol{p}$-space. In each case, all non-empty manifolds are shown.

In each figure it is observed that the coupling coefficient near the tip of the wavevector $\boldsymbol{k}$ is one or two orders of magnitude larger than most other contributions lying on the resonance manifold. Therefore, we might expect that those wave triads with one leg considerably shorter than the other two (as depicted in figure 4*a*) would make a dominant contribution to the overall flux. Indeed, we might further expect that the corresponding wave frequency $\omega_{\boldsymbol{p}}$ – associated with the short leg $\boldsymbol{p}$ of the triad – would also be small, since $p_x \equiv 0$ on the resonance manifold there.

This class of resonant triads, with one short leg and one small frequency, has been identified by McComas & Bretherton (1977) as playing an important role in the spectral transfer of energy in wave turbulence. They named this triad class 'induced diffusion', because the wave-kinetic equation for these triads reduces to a diffusion equation for the wave action. Indeed, for this class of triads the origin of irreversibility is entirely transparent. The large-scale slowly varying short leg $(\boldsymbol{k}', \omega')$ behaves as an adiabatic straining field with respect to the other two members of the triad. In this case, the resonance condition $\delta(\omega + \omega' + \Omega_{\boldsymbol{k}+\boldsymbol{k}'})$ can be approximated as

$$\delta(\omega_{\boldsymbol{k}'} - \boldsymbol{k}' \cdot \boldsymbol{v}_{gr}), \tag{3.48}$$

where the group velocity $\boldsymbol{v}_{gr}$ is $\partial_{\boldsymbol{k}} \Omega_{\boldsymbol{k}}$. Clearly, this is directly analogous to the wave–particle interactions in a Vlasov plasma, with the wave–particle resonance condition

$$\delta(\omega_{\boldsymbol{k}'} - \boldsymbol{k}' \cdot \boldsymbol{v}), \tag{3.49}$$

where, in the case of a Vlasov plasma, $\omega_{\boldsymbol{k}}$ is the frequency of a plasma wave and v is the velocity of the resonant particle. As is well known (Sagdeev & Galeev 1969), when two or more of these wave–particle resonances overlap (quantified by the Chirikov criterion), invariant KAM tori are destroyed and deterministic chaos ensues. Hence, there exists, for the case of the induced diffusion triad class, a rigorous and transparent route to irreversibility in wave interactions, based upon ray chaos.

Finally, we note that triangles with one vertical leg of comparable length to the other two (figure 4*b*) make no contribution to the flux. The coupling coefficient for this triad class, named 'elastic scattering' by McComas & Bretherton, identically vanishes along the circle of radius k centred on the tip of the wavevector $\boldsymbol{k}$, a result supported by the findings of McComas & Bretherton, who noted that, in the absence of vertical asymmetry, such wave interactions will cancel each other out.

4. Discussion

This paper presents calculations which seek to probe and elucidate the fundamentals of the theory of turbulent resistivity in mean field electrodynamics. 'Wavy' MHD in two dimensions with (stable) dispersive linear modes, i.e. two-dimensional MHD with additional body forces (e.g. buoyancy, Coriolis) and evolving scalar fields (e.g. density), was explored as a minimal, yet non-trivial, extension of the simple system originally studied by Cattaneo & Vaĭnshteĭn (1991). The virtue of this system is that it facilitates rigorous, albeit restricted, analytical calculation of the spatial transport of magnetic flux by a spectrum of nonlinearly interacting waves. We again emphasize that here we are concerned with spatial transport of magnetic potential driven by nonlinear wave interactions, rather than spectral transfer.

We have shown that a systematic weak-turbulence expansion of the spatial flux of mean magnetic potential Γ_A in powers of the wave slope $k\tilde{\epsilon}$ yields the result

$$\Gamma_A = -\frac{\partial \langle A \rangle}{\partial z} \left(\overset{(2)}{\eta_T} + \overset{(4)}{\eta_T} + \cdots \right). \tag{4.1}$$

It comes as no surprise that the first term inside the brackets in (4.1) – the second-order (i.e. quasi-linear) contribution to Γ_A – scales as Rm^{-1}, since diffusive dissipation is the only irreversibility available at second order. However, the fourth-order contribution is manifestly independent of Rm, since resonant wave interactions constitute mechanisms of collisionless dissipation which are indeed available at fourth order and higher. Thus, the fourth-order contribution to the turbulent resistivity η_T, although limited in magnitude (by consistency with the usual weak turbulence assumption that $k\tilde{\varepsilon} < 1$), is not quenched at large Rm. Stated explicitly, η_T does not decay asymptotically as Rm^{-1} for asymptotically large Rm (nor any other numbers of the Reynolds type) in the presence of dispersive waves.

A key feature of this calculation is the fact that the fluctuation response time τ_c is not assumed *ab initio* as in EDQNM models of strong turbulence, but explicitly calculated in terms of spectra, wave resonances and spectral auto-correlation times. Taken at face value, these results appear to challenge significantly the theory of Rm-dependent quench of mean field diffusion, as it is understood today. Of course, computational evidence for resistivity quenching stands independently on its own merits, but these results do suggest there are flaws or gaps in our understanding of this important question.

At this point, the pragmatic reader is probably wondering what lessons of broader fluid-dynamic interest are conveyed by this admittedly rather academic study. By way of reply, we note that our findings contain the rather remarkable and counter-intuitive result that, all other factors (such as forcing and visco-resistive dissipation) being equal, the addition of an additional restoring force, such as buoyancy, to the already tightly constrained system of homogeneous high-Rm two-dimensional MHD can actually increase the transport of mean magnetic potential. The result thus uncovers a loophole in the theory of transport and dynamo quenching in large Rm magnetofluids. The loophole identified becomes relevant in the presence of collisionless irreversible evolution via resonant nonlinear wave interactions, a mechanism heretofore not considered in the context of mean field electrodynamics. Thus, we respond to our pragmatic reader with the answer that the lesson learned is that the origins of microscopic irreversibility (be it, for instance, diffusive (e.g. resistive) dissipation, wave–wave resonance, wave–particle or wave–mean resonance) are fundamental to mean field transport processes, since they determine the relevant cross-phase factor which enters the calculation of transport and dynamo coefficients. Note that macroscopic arguments, such as those which invoke the Zel'dovich relation, are intrinsically unsatisfactory, since they do not address the issue of the cross-phase. Indeed, it is instructive to recall that the familiar, oft-quoted Zel'dovich relation, $\eta_T \approx \eta_c \langle b^2 \rangle / \langle B \rangle^2$, is obtained from Ohm's law (i.e. A evolution equation) alone, and so should be insensitive to the addition of buoyancy forces, etc. – a prediction which is clearly in disagreement with the results presented here.

This work also highlights the potentially important role played by dispersive waves which invariably appear in many real geophysical, astrophysical and laboratory magnetofluids. Even if these dispersive waves are confined to a limited range of the fluctuation spectrum and have modest power density, there may still be a significant contribution to the flux of magnetic potential from the wavy range when the Alfvén and intermediate ranges are quenched. Therefore, it would seem inadvisable, when calculating the flux of magnetic potential in real systems, to dismiss *a priori* non-ideal body forces and scalar fields which may give rise to dispersion in the magnetofluid. Rather, such systems must be considered on a case-by-case basis, as is the practice when considering linear and nonlinear wave–particle interactions, and nonlinear wave–wave interactions, in weakly turbulent plasmas (Sagdeev & Galeev 1969).

The broad brush implication of this work is that questions pertinent to the origin and physics of irreversibility assumed in mean field electrodynamics calculations merit far more attention than they usually receive. Computational studies of turbulent resistivity should not merely calculate the scalings of macroscopic relaxation, but also critically test the microscopic elements of the theory. In particular, the Rm dependence of the correlation times and transport cross-phase should be explored. Of course, the results presented here should be tested by high Rm numerical calculations as well. Analytical work should no longer hide behind the unspecified τ_c of EDQNM, but seek to compute response functions and explore their Rm scalings. A full direct interaction approximation calculation of the turbulent resistivity, which self-consistently treats cross-phase, correlation time and spectrum along with η_T, would be interesting, though challenging to actually implement.

More generally, our results suggest that nonlinear wave–wave, wave–particle or wave-flow interaction processes deserve greater attention in the related contexts of mean field electrodynamics and dynamo theory, since they define conceptually transparent, physically sound channels for dissipation and irreversibility at high Rm which are not tied to the microscopic resistivity η_c, and so do not restrict the system's dynamics to slow evolution on time scales set by molecular diffusion.

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Appendix A. Calculation of the elements of the interaction matrix $\mathscr{M}_{\boldsymbol{kpq}}$ from conservation of energy

The coefficients $\mathscr{M}^{\psi ab}_{\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''}$ are determined from conservation of the total energy:

$$E = \sum_{\boldsymbol{k}} k^2 \left\{ |\psi_{\boldsymbol{k}}|^2 + \left|C^{(1)}_{\boldsymbol{k}}\right|^2 + \left|C^{(2)}_{\boldsymbol{k}}\right|^2 \right\}. \tag{A 1}$$

Taking the derivative of (A 1) with respect to time and employing (2.6) we find that

$$\partial_t E = (\partial_t E)_{lin} + (\partial_t E)_{nonlin}. \tag{A 2}$$

We demand that both the linear term and nonlinear terms in (A 2) vanish separately. Thus, $(\partial_t E)_{lin} = 0$ implies that

$$\sum_{\boldsymbol{k}} k^2 \left\{ (\lambda_{\boldsymbol{k}} + \lambda_{-\boldsymbol{k}})|\psi_{\boldsymbol{k}}|^2 + \left(\Omega^{(1)}_{\boldsymbol{k}} + \Omega^{(1)}_{-\boldsymbol{k}}\right)\left|C^{(1)}_{\boldsymbol{k}}\right|^2 + \left(\Omega^{(2)}_{\boldsymbol{k}} + \Omega^{(2)}_{-\boldsymbol{k}}\right)\left|C^{(2)}_{\boldsymbol{k}}\right|^2 \right\} = 0. \tag{A 3}$$

This is satisfied by the conditions

$$\lambda_{-\boldsymbol{k}} = -\lambda_{\boldsymbol{k}}, \quad \Omega^{(j)}_{-\boldsymbol{k}} = -\Omega^{(j)}_{\boldsymbol{k}}. \tag{A 4a, b}$$

Thus, the functions g_1 and g_2, appearing in (2.3), are uniquely determined by the particular form (A 1) for the energy.

The condition that $(\partial_t E)_{nonlin} = 0$ implies

$$\sum_{\boldsymbol{k}} k^2 N^a_{\boldsymbol{k}} x^a_{-\boldsymbol{k}} = 0, \tag{A 5}$$

which, employing the symmetry properties (Krommes 2002)

$$\mathscr{M}^{abc}_{\boldsymbol{kpq}} = \mathscr{M}^{acb}_{\boldsymbol{kqp}} = \mathscr{M}^{abc}_{-\boldsymbol{k},-\boldsymbol{p},-\boldsymbol{q}}, \tag{A 6}$$

yields

$$k^2 \mathscr{M}^{abc}_{\boldsymbol{kpq}} + p^2 \mathscr{M}^{bca}_{\boldsymbol{pqk}} + q^2 \mathscr{M}^{cab}_{\boldsymbol{qkp}} = 0. \tag{A 7}$$

Now, given the form of $\mathscr{M}^{\lambda bc}_{\boldsymbol{kpq}}$ for $\lambda \in \{1, 2\}$ from (2.9), and the fact that

$$\boldsymbol{k} \cdot \boldsymbol{p} \times \hat{\boldsymbol{y}} = \boldsymbol{p} \cdot \boldsymbol{q} \times \hat{\boldsymbol{y}} = \boldsymbol{q} \cdot \boldsymbol{k} \times \hat{\boldsymbol{y}}, \qquad \boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{0}, \tag{A 8a, b}$$

we see that (A 7) is trivially satisfied for the case $a = \lambda$, $b = \mu$ and $c = \nu$ with $\{\lambda, \mu, \nu\} \in \{1, 2\}$.

For the case of $a=b=c=\psi$ we find

$$k^2\mathscr{M}^{\psi\psi\psi}_{\boldsymbol{kpq}} + p^2\mathscr{M}^{\psi\psi\psi}_{\boldsymbol{pqk}} + q^2\mathscr{M}^{\psi\psi\psi}_{\boldsymbol{qkp}} = 0, \tag{A 9}$$

which is satisfied by

$$\mathscr{M}^{\psi\psi\psi}_{\boldsymbol{kpq}} = (\boldsymbol{k}\cdot\boldsymbol{p}\times\hat{\boldsymbol{y}})\frac{p^2-q^2}{k^2}. \tag{A 10}$$

Equation (A 10) can be verified directly from the Euler equation, because the ψ-nonlinearity comes from the advection term

$$\frac{1}{k^2}(\boldsymbol{v}\cdot\nabla\nabla^2\psi)_{\boldsymbol{k}} = \sum_{\boldsymbol{p}+\boldsymbol{q}=-\boldsymbol{k}} (\boldsymbol{k}\cdot\boldsymbol{p}\times\hat{\boldsymbol{y}})\frac{p^2-q^2}{k^2}\psi_{-\boldsymbol{p}}\psi_{-\boldsymbol{q}}. \tag{A 11}$$

The case of $a=\psi$, $b=\mu$, $c=\nu$, $\{\mu,\nu\}\in\{1,2\}$ implies that

$$\mathscr{M}^{\psi\mu\nu}_{\boldsymbol{kpq}} = -(\boldsymbol{k}\cdot\boldsymbol{p}\times\hat{\boldsymbol{y}})\left\{\frac{p^2g_\mu^2(\boldsymbol{p})-q^2g_\mu^2(\boldsymbol{q})}{k^2g_\mu(\boldsymbol{p})g_\mu(\boldsymbol{q})}\right\}\delta^{\mu\nu}. \tag{A 12}$$

Again, the interaction matrix given by (A 12) can be verified for the case of the A-nonlinearity, which, in the case of two-dimensional MHD, comes from the $\boldsymbol{j}\times\boldsymbol{b}$ force:

$$-\frac{1}{k^2}(\boldsymbol{b}\cdot\nabla\nabla^2 A)_{\boldsymbol{k}} = \sum_{\boldsymbol{p}+\boldsymbol{q}=-\boldsymbol{k}} (\boldsymbol{k}\cdot\boldsymbol{p}\times\hat{\boldsymbol{y}})\,\frac{q^2-p^2}{k^2}A_{-\boldsymbol{p}}A_{-\boldsymbol{q}}, \tag{A 13}$$

where we have used $C_{1,\boldsymbol{k}}\equiv A_{\boldsymbol{k}}$ and $g_1(\boldsymbol{k})\equiv 1$.

The final case, that of $a=b=\psi$, $c=\nu\in\{1,2\}$, is trivially satisfied.

Appendix B. Derivation of linear damping rates and second-order turbulent resistivities for two-dimensional MHD, β-plane MHD and stratified MHD

We now calculate the linear damping rates and second-order turbulent resistivities for a spectrum of (i) Alfvén waves, (ii) Rossby–Alfvén waves, and (iii) magneto-internal waves.

B.1. *Alfvén waves*

The linearized equations of motion for two-dimensional MHD are

$$(-\mathrm{i}\omega+\nu k^2)\psi_{\boldsymbol{k}\omega} = \mathrm{i}B_0k_xA_{\boldsymbol{k}\omega}, \tag{B 1a}$$

$$(-\mathrm{i}\omega+\eta_c k^2)A_{\boldsymbol{k}\omega} = \mathrm{i}B_0k_x\psi_{\boldsymbol{k}\omega}. \tag{B 1b}$$

The dispersion relation for (B 1*a*, *b*) is

$$(-\mathrm{i}\omega+\nu k^2)(-\mathrm{i}\omega+\eta_c k^2)+B_0^2k_x^2 = 0. \tag{B 2}$$

We separate the frequency ω into a real and imaginary part $\omega=\omega_{\boldsymbol{k}}+\mathrm{i}\gamma_{\boldsymbol{k}}$ where, again, $\gamma_{\boldsymbol{k}}$ is the linear growth rate (negative for damping). Taking the real part of (B 2) yields the usual solutions $\omega_{\boldsymbol{k}}=\pm B_0k_x$. The imaginary part of (B 2), neglecting terms which are quadratic in $\gamma_{\boldsymbol{k}}$, νk^2, $\eta_c k^2$ and $\mathscr{D}k^2$, gives the damping rate as

$$\gamma_{\boldsymbol{k}} = -\frac{\nu+\eta_c}{2}k^2, \tag{B 3}$$

as required. Therefore, the turbulent resistivity is, from (3.24),

$$\overset{(2)}{\eta_T} \approx \sum_{\boldsymbol{k}} \frac{|\eta_c-\nu|\,k^2}{2\omega_{\boldsymbol{k}}^2}\langle v^2\rangle_{\boldsymbol{k}\omega}. \tag{B 4}$$

B.2. *Rossby–Alfvén waves*

The linearized equations of motion for β-plane MHD are

$$(-\mathrm{i}\omega - \mathrm{i}\beta k_x/k^2 + \nu k^2)\psi_{\boldsymbol{k}\omega} = \mathrm{i}B_0 k_x A_{\boldsymbol{k}\omega}, \tag{B 5a}$$

$$(-\mathrm{i}\omega + \eta_c k^2)A_{\boldsymbol{k}\omega} = \mathrm{i}B_0 k_x \psi_{\boldsymbol{k}\omega}, \tag{B 5b}$$

with the dispersion relation

$$(-\mathrm{i}\omega - \mathrm{i}\beta k_x/k^2 + \nu k^2)(-\mathrm{i}\omega + \eta_c k^2) + B_0^2 k_x^2 = 0. \tag{B 6}$$

Again writing ω in terms of a real and imaginary part, and discarding terms quadratic in the damping and diffusion rates, we find, for the real part of (B 6), the usual dispersion relation for Rossby–Alfvén waves

$$\omega_{\boldsymbol{k}}^2 + \omega_{\boldsymbol{k}} \frac{\beta k_x}{k^2} - B_0^2 k_x^2 = 0. \tag{B 7}$$

The imaginary part of (B 6) gives the equation for the damping rate $\gamma_{\boldsymbol{k}} = \mathrm{Im}\,\omega$ as

$$\gamma_{\boldsymbol{k}} \left(\omega_{\boldsymbol{k}} + \frac{\beta k_x}{2k^2} \right) = -\frac{\nu + \eta_c}{2} k^2 \omega_{\boldsymbol{k}} + \eta_c k^2 \frac{\beta k_x}{2k^2}. \tag{B 8}$$

On small scales $\ell \ll \ell^* = \sqrt{B_0/\beta}$, the linear frequency $\omega_{\boldsymbol{k}}$ is approximately $\pm B_0 k_x$ so that (B 8) reduces to (B 3), the damping rate obtained for a spectrum of Alvén waves. Likewise, the associated turbulent resistivity to second order will be given by (B 4). On large scales $\ell \gg \ell^* = \sqrt{B_0/\beta}$, the velocity and magnetic fields are decoupled, with linear frequencies of approximately $\beta k_x/k^2$ and 0, respectively. The velocity fluctuations are damped according to

$$\gamma_{\boldsymbol{k}} \approx -\nu k^2, \tag{B 9}$$

while the magnetic fluctuations are damped as

$$\gamma_{\boldsymbol{k}} \approx -\eta_c k^2. \tag{B 10}$$

The turbulent resistivity, for the finite-frequency velocity fluctuations, is

$$\overset{(2)}{\eta_T} \approx \sum_{\boldsymbol{k}} \frac{|\eta_c - \nu|\, k^2}{\omega_{\boldsymbol{k}}^2} \langle v^2 \rangle_{\boldsymbol{k}\omega}. \tag{B 11}$$

Notice that $\omega_{\boldsymbol{k}}$ is now the Rossby wave frequency $\beta k_x/k^2$.

B.3. *Magneto-internal waves*

The linearized equations of motion for stratified MHD are

$$(-\mathrm{i}\omega + \nu k^2)\psi_{\boldsymbol{k}\omega} = \mathrm{i}B_0 k_x A_{\boldsymbol{k}\omega} + \mathrm{i}N k_x/k^2 \chi_{\boldsymbol{k}\omega}, \tag{B 12a}$$

$$(-\mathrm{i}\omega + \eta_c k^2)A_{\boldsymbol{k}\omega} = \mathrm{i}B_0 k_x \psi_{\boldsymbol{k}\omega}, \tag{B 12b}$$

$$(-\mathrm{i}\omega + \mathscr{D} k^2)\chi_{\boldsymbol{k}\omega} = \mathrm{i}N k_x \psi_{\boldsymbol{k}\omega}, \tag{B 12c}$$

with the dispersion relation

$$(-\mathrm{i}\omega+\nu k^2)(-\mathrm{i}\omega+\eta_c k^2)(-\mathrm{i}\omega+\mathscr{D}k^2)+B_0^2 k_x^2(-\mathrm{i}\omega+\mathscr{D}k^2)+\frac{N^2 k_x^2}{k^2}(-\mathrm{i}\omega+\eta_c k^2) = 0. \tag{B 13}$$

Again, the real part of (B 13) yields the usual magento-internal wave dispersion relation $\omega_{\boldsymbol{k}}^2 = \varOmega_{\boldsymbol{k}}^2 = B_0^2 k_x^2 + N^2 k_x^2/k^2$, as well as the zero-frequency mode $\omega_{\boldsymbol{k}} \equiv 0$. The imaginary part of (B 13) gives the following equation for the growth rate $\gamma_{\boldsymbol{k}} = \mathrm{Im}\,\omega$:

$$\left(3\omega_{\boldsymbol{k}}^2 - \Omega_{\boldsymbol{k}}^2\right)\gamma_{\boldsymbol{k}} = -\omega_{\boldsymbol{k}}^2(\nu + \eta_c + \mathscr{D})\,k^2 + B_0^2 k_x^2 \mathscr{D} k^2 + \frac{N^2 k_x^2}{k^2}\eta_c k^2. \tag{B 14}$$

Again, we consider small ($\ell \ll \ell^* = B_0/N$) and large ($\ell \gg \ell^*$) scales separately. On small scales, the density field ($\omega_{\boldsymbol{k}} \equiv 0$) is decoupled from the other two $\omega_{\boldsymbol{k}} = \pm B_0 k_x$ and damps as

$$\gamma_{\boldsymbol{k}} \approx -\mathscr{D}k^2, \tag{B 15}$$

while the velocity and magnetic fields oscillate as Alfvén waves and are damped according to (B 3). Likewise, the turbulent resistivity to second order will be that for a spectrum of Alfvén waves.

For large scales, the magnetic field ($\omega_{\boldsymbol{k}} \equiv 0$) is decoupled from the velocity and density fields, which oscillate as internal waves ($\omega_{\boldsymbol{k}} = \pm\, N k_x/|\boldsymbol{k}|$). The magnetic fluctuations will decay as

$$\gamma_{\boldsymbol{k}} \approx -\eta_c k^2, \tag{B 16}$$

while the velocity and density fields are damped according to

$$\gamma_{\boldsymbol{k}} \approx -\frac{\nu + \mathscr{D}}{2}k^2. \tag{B 17}$$

The turbulent resistivity associated with a spectrum of internal gravity waves is then

$$\overset{(2)}{\eta_T} \approx \sum_{\boldsymbol{k}} \frac{|2\eta_c - \nu - \mathscr{D}|\,k^2}{2\omega_{\boldsymbol{k}^2}}\langle v^2\rangle_{\boldsymbol{k}\omega}. \tag{B 18}$$

Again, the frequency $\omega_{\boldsymbol{k}}$ appearing in (B 18) is that of an internal gravity wave Nk_x/k^2.

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